OXFORD IB DIPLOMA PROGRAMME

# MIXED REVIEW

# MATHEMATICS:
## APPLICATIONS AND INTERPRETATION

HIGHER LEVEL

COURSE COMPANION

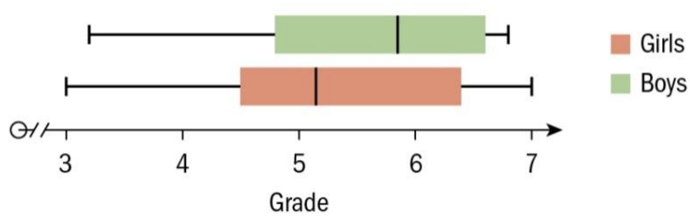
ENHANCED ONLINE

Suzanne Doering
Panayiotis Economopoulos
Peter Gray
David Harris
Tony Halsey
Michael Ortman
Nuriye Sirinoglu Singh
Jennifer Chang Wathall

OXFORD

# 2 Representing and describing data: descriptive statistics

**1** The box-whisker plots below illustrate the grades for boys and girls in class 2




For each of the questions below, answer "boys", "girls" or "not possible to tell"

**a** Which sex had the greater range of grades?

**b** Which sex had the higher median grade?

**c** Which sex had the higher mean grade?

**d** Which sex had the lowest grade?

**e** Which sex had the most students over 5.0?

**f** Were there more boys or more girls?

**2** In this question all final answers should be given correct to three significant figures
The volume of a hemisphere is 1 litre.

**a** Find the radius of the hemisphere.

**b** Find the surface area of the hemisphere.

A cone has the same base radius as the hemisphere, and the same volume.

**c** Find the height of the cone.

**3** A survey was conducted of the number of bedrooms in 208 randomly chosen houses. The results are shown in the following table.

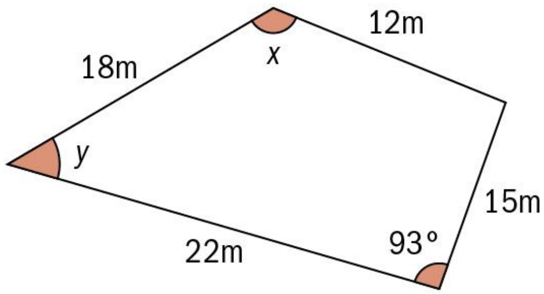
| | | | | | | |
|---|---|---|---|---|---|---|
| Number of bedrooms | 1 | 2 | 3 | 4 | 5 | 6 |
| Number of houses | 41 | 60 | 52 | 32 | 15 | 8 |

**a** Find the mean number of bedrooms per house.

**b** Find the standard deviation of the number of bedrooms per house.

**c** Find how many houses have a number of bedrooms greater than one standard deviation above the mean.

**4** Find $x$ and $y$ in the following diagram:




**5** A ship starts from a harbour, travels for 45 km on a bearing of 050º and then turns to travel a further 65 km on a bearing of 290º.

**a** Draw a diagram to illustrate the journey, marking clearly the known angles.

**b** How far is the ship from the harbour at this point?

**c** What bearing will the ship need to take in order to return to the harbour?

**Exam-style questions**

**6** The owner of a wildlife sanctuary wishes to collect data about the different species of butterfly in their butterfly house. They decide to collect a sample of 10 butterflies from three different species.

**a** State the most appropriate sampling technique the owner should use, giving reasons, and suggest two advantages and one disadvantage of this sampling technique. (4)

Below are the wingspans, in cm, of 10 Postman Butterflies.

7.1, 6.8, 5.9, 6.7, 6.6, 5.6, 8.9, 7.5, 7.0, 7.3

**b** Find the median and interquartile range. (5)

**c** Deduce if there are any outliers and draw a box and whisker diagram to represent the data. (6)

**7** A coffee plantation owner wishes to monitor the rainfall at two of her plantations in different areas of Brazil.

The average rainfall, in mm, for the two locations for a period of 6 months is give in the table.

| | July | August | September | October | November | December |
|---|---|---|---|---|---|---|
| Sao Paulo | 44 | 39 | 81 | 124 | 146 | 201 |
| Minas Gerais | 12 | 16 | 37 | 103 | 202 | 198 |

**a** By calculating (to two decimal places) the mean and standard deviation of the rainfall in the two different regions, compare the differences in rainfall. (10)

**b** The plantation owner is concerned about extreme weather affecting her crops. Use your values from part **a** to comment on her concerns. (2)

**8** Charlotte wants to analyse her walking habits over a period of 120 days.

The grouped frequency table shows the daily walking distances, in metres, registered on her pedometer.

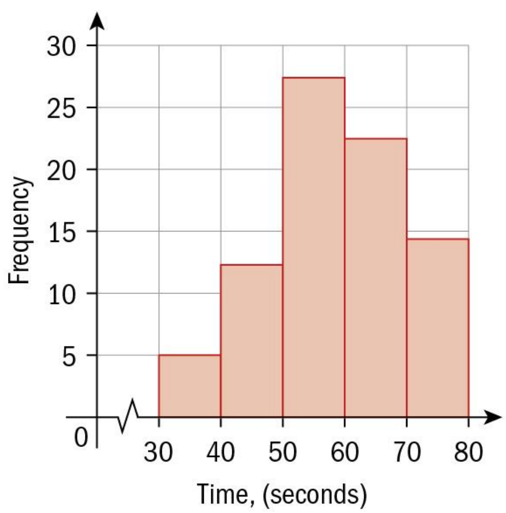
| Daily distance, *d* (m) | Frequency |
|---|---|
| $0 \le d < 1000$ | 3 |
| $1000 \le d < 2000$ | 11 |
| $2000 \le d < 3000$ | 15 |
| $3000 \le d < 4000$ | 21 |
| $4000 \le d < 5000$ | 32 |
| $5000 \le d < 6000$ | 15 |
| $6000 \le d < 7000$ | 21 |
| $7000 \le d < 8000$ | 1 |
| $8000 \le d < 9000$ | 1 |

**a** Find estimates for the mean, variance and standard deviation. (Give your answers to 2 decimal places) (5)

**b** Martin also collected data on his walking habits for the same period of time. His mean distance was 3546 m and his standard deviation was 986 m. Compare the walking habits of Charlotte and Martin. (2)

**c** Martin remembered that he normally walked 450 m each morning before switching on his pedometer. All of his distances should have 450 metres added to them. He also wanted to convert from metres to kilometres. Calculate Martin's new mean and standard deviation, in kilometres. (4)

**9** The frequency histogram represents the time taken for 80 people to complete a logic puzzle, in seconds.




**a** Use the frequency histogram to construct a cumulative frequency curve for this data. (4)

**b** Find the median, lower quartile and upper quartile for this data. Hence, plot a box and whisker diagram, making sure you consider any outliers. (12)

**c** The organiser of the logic puzzle claims that 10% of people can complete the puzzle in under 40 seconds. Find the $10^{th}$ percentile and comment on their claim. (3)

**10** A seaside town collected some data over a period of 8 years. The table below shows the average daily temperature, °C, over July and August along with the number of jellyfish stings treated by the life guards and the average daily ice cream sale by the beach café.

| | 2010 | 2011 | 2012 | 2013 | 2014 | 2015 | 2016 | 2017 |
|---|---|---|---|---|---|---|---|---|
| Ave. temp (°C) | 22.5 | 26.1 | 25.2 | 23.7 | 27.8 | 25.6 | 29.2 | 28.7 |
| jellyfish stings treated | 13 | 14 | 15 | 12 | 29 | 22 | 45 | 37 |
| Average ice cream sales | 287 | 427 | 392 | 314 | 463 | 407 | 568 | 552 |

**a** Calculate Pearson's Product-Moment Correlation Coefficient (to 3 d.p) and comment on the correlation for

**i** average temperature and jellyfish stings

**ii** average temperature and average ice cream sales

**iii** jellyfish stings and average ice cream sales. (9)

**b** Comment on whether the number of jellyfish stings affects the number of Ice Cream sales. (2)

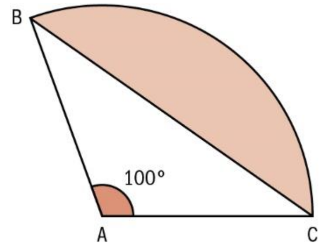
**11** A sphere has a radius 5 cm.

**a** Use 3.14 as an approximation to π to calculate an estimate of the surface area of the sphere. (2)

**b** Find the exact value of the sphere's surface area. Give your answer as a multiple of π.(2)

**c** Find the percentage error of your approximation from part *a*, giving you answer in standard form correct to 2 decimal places. (3)

**12** The sector ABC has an angle 100° and an area of 42.76 cm$^2$.




**a** Calculate the radius of the sector. Give your answer to 2 decimal places. (3)

**b** Calculate the length of the chord BC. Give your answer to 2 decimal places. (3)

**c** Find the area of the of the shaded segment that is between the chord BC and the arc BC. Give your answer to 2 decimal places. (3)

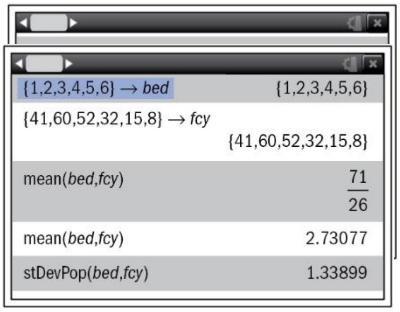
**Answers**

**1** **a** girls **b** boys **c** not possible to tell **d** girls
**e** not possible to tell **f** not possible to tell

**2** **a** $\frac{2}{3}\pi r^3 = 1000 \Rightarrow r = 7.82$ cm

**b** $2\pi r^2 + \pi r^2 = 3\pi r^2 = 576$ cm$^2$

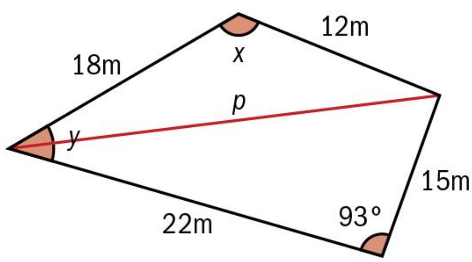
**c** $\frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3 \Rightarrow h = 2r = 15.6$ cm




**3** **a, b**

**c** $2.73 + 1.34 = 4.07$ and there are 15 + 8 = 23 houses with more than 4.07 bedrooms.

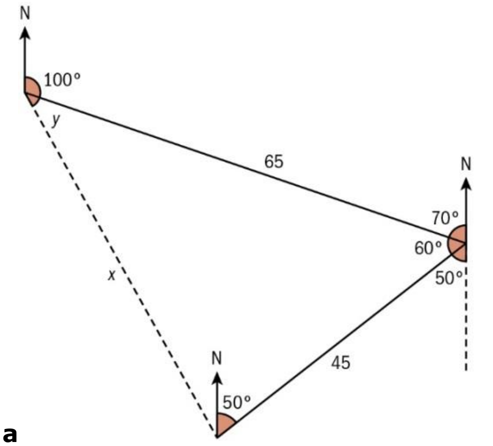
**4**




$p^2 = 15^2 + 22^2 - 2.22.15\cos(93^\circ) \Rightarrow p = 27.3$ m

$\cos x = \frac{12^2 + 18^2 - p^2}{2.12.18} = -0.638 \Rightarrow x = 130^\circ$

$y = \cos^{-1}\left(\frac{18^2 + p^2 - 12^2}{2.18.p}\right) + \cos^{-1}\left(\frac{22^2 + p^2 - 15^2}{2.22.p}\right) = 53.1^\circ$




**5** **a**

**b** $x^2 = 45^2 + 65^2 - 2.45.65.\cos 60^\circ \Rightarrow x = 57.7\text{ km}$

**c** $\dfrac{x}{\sin 60^\circ} = \dfrac{45}{\sin y} \Rightarrow y = 42.5$ so bearing is 153°

**6** **a** Quota (A1)

because this is non- random sampling (R1)

Advantages – any 2 of – inexpensive, easy to perform, quick (R2)

Disadvantage – any 1 of – biased, unreliable (R1)

**b** Put in order M1

Median = 6.9 A1

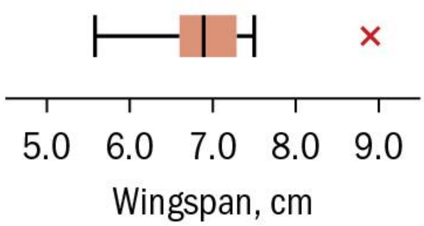
LQ = 6.6 A1

UQ = 7.3 A1

IQR = 0.7 A1

**c** 6.6 - 1.5 x 0.7 or 7.3 + 1.5 x 0.7 M1

identify 8.9 as outlier A1




Cross at outlier A1

3 vertical lines correct A1

All vertical lines correct A1

Box correct A1

**7** **a** Sao Paulo: Mean = 105.83 mm M1A1, SD = 57.6 mm M1A1

Minas Gerais: Mean = 94.6 mm M1A1, SD = 80.22 mm M1A1

Mean lower for Minas Gerais so less rainfall R1

SD high for Minas Gerais so rainfall less consistent R1

**b** Sensible comment about the means R1

Sensible comment about the SD R1

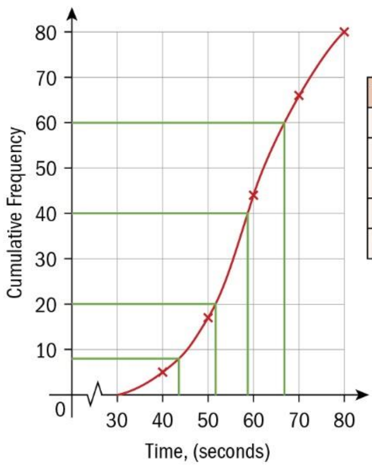
e.g.

*The mean rainfall for the two regions are not hugely different but if less rain is better for crops Minas Gerais may do better. However, the SD is larger for Minas Gerais so they may have more extreme weather so could be bad for the crops.*

**8** **a** Mean = 4233.33 m (M1)A1

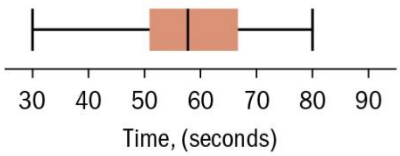
Variance = 2 862 222.22 $\text{m}^2$ (M1)A1

| | | |
|---|---|---|
| | SD = 1691.81 m | A1 |
| **b** | Martin's mean is lower so suggests he walks less | R1 |
| | Charlotte's SD is larger so her distances are more varied | R1 |
| **c** | Mean = $\frac{3546+450}{1000}$ = 3.996 km | M1A1 |
| | SD = $\frac{986}{1000}$ = 0.986 km | M1A1 |
| **9 a** | Correct table (can be implied by correct curve) | (M1A1) |
| | Correct curve (point plotted on upper class boundaries) | M1A1 |




| Time, S | CF |
|---|---|
| S < 40 | 5 |
| S < 50 | 17 |
| S < 60 | 44 |
| S < 70 | 66 |
| S < 80 | 80 |

| | | |
|---|---|---|
| **b** | Median = 58 s | M1A1 |
| | LQ = 51 s | M1A1 |
| | UQ = 67 s | M1A1 |
| | IQR = 16 s | A1 |
| | 51 − 1.5 x 16 or 67 + 1.5 x 16 | M1 |
| | No outliers | R1 |




| | | | | |
|---|---|---|---|---|
| | 3 vertical lines correct | A1 | All vertical lines correct | A1 |
| | Box correct | | | A1 |
| **c** | $10^{th}$ percentile = 43 | | | M1A1 |
| | does not agree with claim | | | R1 |
| **10 a i** | 0.899 | (M1)A1 | Strong positive correlation | R1 |
| **ii** | 0.987 | (M1)A1 | (Very) strong positive correlation | R1 |

**iii** 0.916 (M1)A1 strong positive correlation R1

**b** The PMCC shows there is a positive correlation between jellyfish stings and ice cream sales. However, correlation does not imply causation. (R1)

An increase in both the number of jellyfish stings, and the number of ice cream sales, is likely to be caused by a rise in the temperature. (R1)

**11 a** $4 \times 3.14 \times 5^2 = 314\,cm^2$ M1A1

**b** $4 \times \pi \times 5^2 = 100\pi\,cm^2$ M1A1

**c** $\frac{100\pi - 314}{100\pi} \times 100$ M1

0.0507% M1

$5.07 \times 10^{-2}$ % A1

**12 a** $\frac{100}{360} \times \pi \times r^2 = 42.76$ M1A1

$r^2 = 48.999...$

7.00 cm A1

**b** $(BC)^2 = 7^2 + 7^2 - 2 \times 7 \times 7 \times \cos 100$ M1A1

$(BC)^2 = 115.0175...$

$BC = 10.72$ cm A1

**c** Area triangle $ABC$ = $\frac{1}{2} \times 7 \times 7 \times \sin 100 = 24.1277...$ M1

Area shaded = $42.76 - (\frac{1}{2} \times 7 \times 7 \times \sin 100)$ M1

18.63 $cm^2$ A1

# 3 Dividing up space: coordinate geometry, lines, Voronoi diagrams, vectors

**1** For the two points $A(1,2)$ and $B(-2,4)$

**a** Find AB

**b** Find the mid-point of [AB]

**c** Find the equation of (AB)

**2** Two hikers are walking towards each other along a straight path. At 11.00 the first is 150m east, 200m north, and 40m higher than a nearby mountain hut. The other hiker is 250m east, 150 south and 70m higher than the hut.

**a** Write down the coordinates of each hiker taking the mountain hut as the origin of the coordinate system.

**b** Assuming the hikers are both walking at the same speed find the coordinates of the point at which they will meet.

**c** If they meet at 11.06, find how fast they are walking.

**3** Three rescue helicopters need to cover a region of a National Park form their three bases within the region. When an accident occurs, whichever helicopter is closest will fly to the scene.

The region can be thought of as a rectangular coordinate system with $0 \le x \le 9$ and $0 \le y \le 6$. The three bases for the helicopters are A $(2,4)$, B $(6,4)$ and C $(6,1)$. Where the measurement are in 10 km units.

**a** Draw a Voronoi diagram showing the regions covered by each of the helicopters.

**b** Calculate the area of each region covered by the helicopters.

**4** A town has 4 large supermarkets, A,B,C and D, whose positions can be given as $A(2,5)$, B $(7,5)$, $C(4,2)$ and $D(7,1)$. The area of the town can be taken as the area within the polygon formed by the four supermarkets.

The on the next page shows the four supermarkets and the boundary of the town. The perpendicular bisectors have also been added.

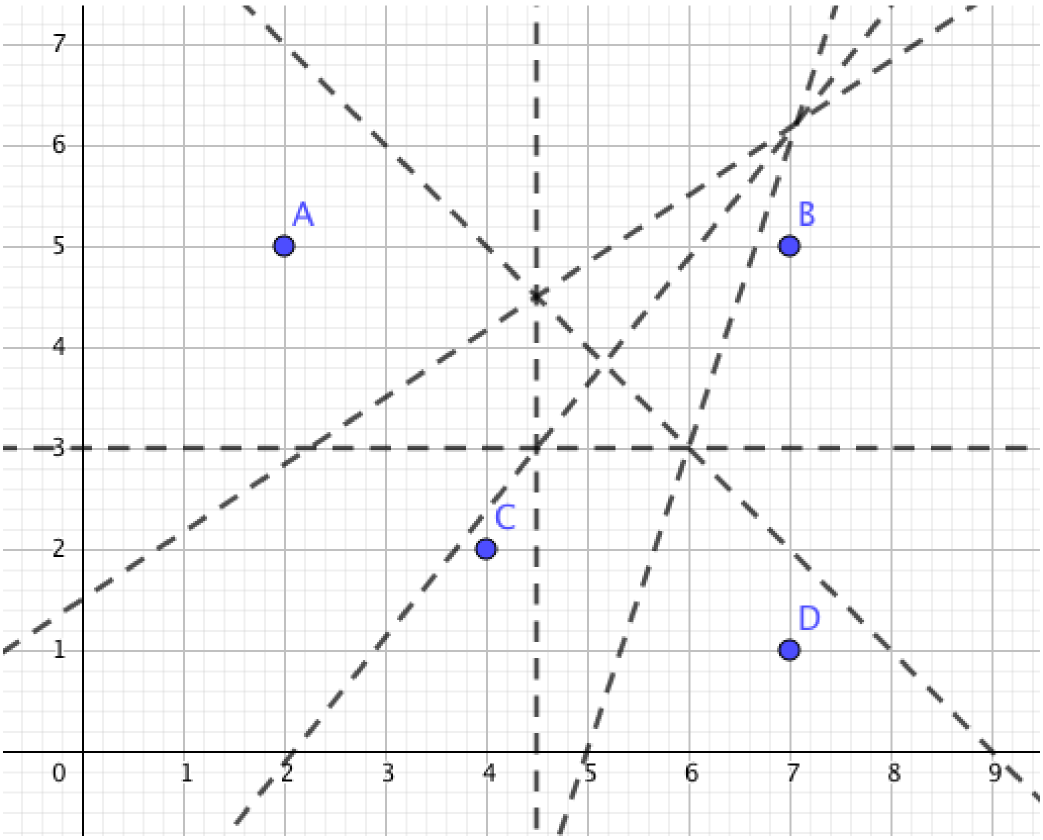
**a** On the diagram, use the incremental algorithm to draw the Voronoi diagram showing the regions closest to each supermarket.

A fifth supermarket is to open inside the town.

**b** Calculate the coordinates of the point where the supermarket should open if it is to be as far as possible from the existing supermarkets.

**c** Find the minimum distance of the new supermarket from the existing supermarkets.

This diagram is for question 4:




**5** A ship's progress is been tracked by a coastguard. He records the position of the ship at 12 noon to be $(12,8)$ relative to the position of the coastguard station. The measurements are in kilometers with the $x$-axis due east and the $y$-axis due north.

At 12.15 the position of the ship is measured at $(9,12)$.

**a** Find **i** the velocity **ii** the speed of the ship

**b** Write down the equation that gives the ship's displacement $t$ hours after 12 noon.

A lighthouse is situated at the point with coordinates $(3,20)$

**c** Show that if it carries on with the same course the ship will collide with the lighthouse.

At 12.15 the ship alters direction so it is travelling with a velocity of 24 kmh$^{-1}$ north-west.

**d** Find the new velocity of the ship.

**e** Write down the equation for the displacement of the ship at time $t$ for $t \geq 0.25$

**f** Find the displacement of the ship from the lighthouse at time $t$, $t \geq 0.25$

**g** Find the minimum distance of the ship from the lighthouse.

**6** An aircraft (A) is taking off from an airport at the same time as another (B) is coming in to land.

The displacement of the first aircraft (in kilometres) can be given by the equation $r = t\begin{pmatrix}35\\50\\14\end{pmatrix}$

and the second as $r = \begin{pmatrix}2000\\1500\\720\end{pmatrix} + t\begin{pmatrix}-45\\-30\\-10\end{pmatrix}$

The $x$-axis is due east and the $y$-axis due north

**a** Find the value of $t$ at which the two aircraft have the same height

**b** At this height find

**i** the vector $\overrightarrow{BA}$

**ii** the bearing of A from B

**iii** the distance the two aircraft are apart.

**c** Find the time $t$ at which B will land.

**7** A satellite records the coordinates of the vertices of a triangular wood as $(20, 43, 16)$, $(28, 32, 18)$ and $(26, 38, 19)$ relative to a fixed point.

Given that the coordinates are given in kilometres find the area of the wood.

**8** Two machines are digging the level section of a tunnel from opposite ends. The displacement of the first machine is given by $\mathbf{r} = t\begin{pmatrix}2.1\\3.2\\0\end{pmatrix}$ where $t$ is the time in hours from when it began work on the level section and distances are given in metres.

With respect to the same origin the position of the second machine as it began work on the level section is given by $\mathbf{r} = \begin{pmatrix}84\\128\\0\end{pmatrix} + t\begin{pmatrix}-3.36\\-5.12\\0\end{pmatrix}$

**a** Find the length of the level section of the tunnel.

**b** Verify that both machines are drilling in the same direction.

**c** How many hours after the drilling begins will the two machines meet?

**9** At 14.00 a ship is sighted at a position 20km from a coastguard station on a bearing of $330°$.

**a** Find the coordinates of the ship's position at 14.00 taking the coastguard station as the origin, the $x$-axis as east and the $y$-axis as north.

The ship is travelling due east and at 14.30 it is due north of the coastguard station.

**b** Write down the equation for the displacement of the ship at time $t$ hours after 14.00.

At 16.00 the coastguard sends out a pilot boat to intercept the ship. The boat leaves the coastguard station on a bearing of $045°$

**c** Given that the boat intercepts the ship without the need to change direction find the time at which this occurs.

**d** Find the equation for the displacement of the boat at time $t$ ($t \geq 2$) in terms of $v$ the speed the boat was travelling

**e** Find the speed the boat was travelling.

**Exam-style questions**

**10** A flock of geese is flying due South with a speed of 80 km $h^{-1}$. Find the speed and the bearing of the flock of geese if the wind starts blowing

**a** from the South at speed 20 km $h^{-1}$ (1)

**b** from the South-West at speed 35 km $h^{-1}$. (4)

**11** A set of integers comprises the numbers $1, 2, 5, 6, a, b$, where $a < b$ and $a, b \in \mathbb{N}$.

The mean and the median of the set of numbers are both equal to 5.

Find the value of $a$ and the value of $b$. (5)

**12** Three lines with line equations $6x - 3y = 1$, $x - 2y = 2$ and $3x + y = 1$ intersect at points $A$, $B$, and $C$. Find the area of the triangle $ABC$. (8)

**13** Points $A(1, 2)$, $B(3, 1)$, $C(5, 5)$ lie on the circumference of a circle.

Use the fact that the perpendicular bisectors of any two chords will intersect at the centre of the circle to find the radius of the circle. (8)

**14** 'Skewness' in data indicates a lack of symmetry in the distribution of a data set.

One quantity that is used to measure skewness is $\frac{3(\text{mean} - \text{median})}{\text{standard deviation}}$. The greater the value of this quantity, the greater the skewness the data is said to have.

The following data indicates the ages of patients in the ward of a hospital.

23, 26, 30, 32, 48, 55, 56, 60, 62, 70, 74, 74, 76, 80, 88

Using the measure given above, calculate the skewness value for this set of data. Comment on your result. (8)

**Answers**

**1** **a** 3.61

**b** $(-0.5, 3)$

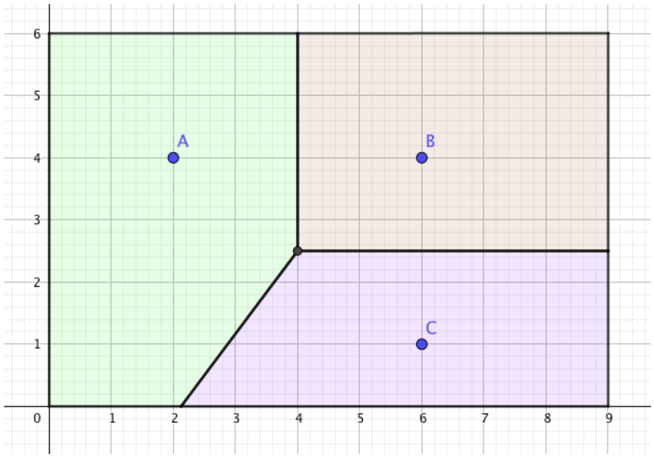
**c** $y = -\frac{2}{3}x + \frac{8}{3}$ or $3y + 2x - 8 = 0$

**2** **a** $(150, 200, 40)$ and $(250, -150, 70)$

**b** $(200, 25, 55)$

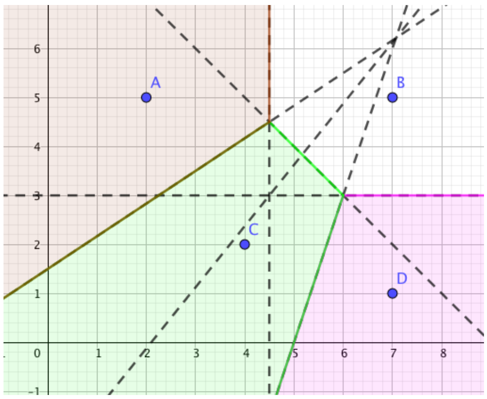
**c** 1.83 $\text{kmh}^{-1}$

**3** **a**




**b** A 2166$\text{km}^2$ B 1750$\text{km}^2$ C 1484 $\text{km}^2$

**4** **a**




**b** $(6, 3)$ **c** 2.24

**5** **a** **i** $\begin{pmatrix} -12 \\ 16 \end{pmatrix}$ **ii** 20$\text{kmh}^{-1}$

**b** $\mathbf{r} = \begin{pmatrix} 12 \\ 8 \end{pmatrix} + t\begin{pmatrix} -12 \\ 16 \end{pmatrix}$

**c** when $t = 0.75$ the position of the ship is $(3, 20)$

**d** $\begin{pmatrix} -17.0 \\ 17.0 \end{pmatrix}$

**e** $r = \begin{pmatrix} 9 \\ 12 \end{pmatrix} + (t - 0.25)\begin{pmatrix} -17 \\ 17 \end{pmatrix} = \begin{pmatrix} 13.25 \\ 7.75 \end{pmatrix} + t\begin{pmatrix} -17 \\ 17 \end{pmatrix}$

**f** $\begin{pmatrix} 10.25 - 17t \\ -12.25 + 17t \end{pmatrix}$

**g** 1.4 km

**6** **a** 30 s

**b** **i** $\begin{pmatrix} 400 \\ 900 \\ 0 \end{pmatrix}$ **ii** $024^\circ$ **iii** 985 m

**c** $t = 72$ s

**7** 18.364 km$^2$

**8** **a** 153 m

**b** $-1.6\begin{pmatrix} 2.1 \\ 3.2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3.36 \\ -5.12 \\ 0 \end{pmatrix}$

**c** 15.38 hours

**9** **a** $(-10, 17.3)$

**b** $r = \begin{pmatrix} -10 \\ 17.3 \end{pmatrix} + t\begin{pmatrix} 20 \\ 0 \end{pmatrix}$

**c** $-10 + 10t = 17.3$ so $t = 2.73$

**d** $r = (t - 2)\begin{pmatrix} 0.707v \\ 0.707v \end{pmatrix}$

**e** 33.5 kmh$^{-1}$

**10** **a** $v = 80 - 20 = 60$ km h$^{-1}$ with the bearing of $180^\circ$. A1

**b** Using the cosine rule the speed is
$v^2 = 80^2 + 35^2 - 2 \times 80 \times 35\cos 45^\circ = 5175 - 2800\sqrt{2} \approx 1215.2$, $v \approx 34.9$ kmh$^{-1}$. M1A1

We can find the bearing using the sine rule $\dfrac{35}{\sin\alpha} = \dfrac{34.86}{\sin 45^\circ} \Rightarrow \sin\alpha = 0.710$, and

so $\alpha \approx 45.2^\circ$. The bearing is $180^\circ - \alpha \approx 134.8^\circ$. M1A1

**11** $\dfrac{1 + 2 + 5 + 6 + a + b}{6} = 5$ M1A1

$a + b = 16$

Since Median = 5, when numbers are arranged in ascending order, we must have $\frac{3^{rd} \text{ number} + 4^{th} \text{ number}}{2} = 5$. Hence, it must be true that $a = 5$. R1

Therefore $a = 5$ and $b = 11$ A1A1

**12** The lines $6x - 3y = 1 \Rightarrow y = 2x - \frac{1}{3}$ and $x - 2y = 2 \Rightarrow y = \frac{1}{2}x - 1$ intersect at $2x - \frac{1}{3} = \frac{1}{2}x - 1 \Rightarrow x = -\frac{4}{9}$ and $y = -\frac{11}{9}$ . M1A1

The third line $3x + y = 1 \Rightarrow y = 1 - 3x$ intersects the first line where $1 - 3x = 2x - \frac{1}{3} \Rightarrow x = \frac{4}{15}, y = \frac{1}{5}$ A1

The third line intersects the second line where $1 - 3x = \frac{1}{2}x - 1 \Rightarrow x = \frac{4}{7}, y = -\frac{5}{7}$. A1

Suppose $A = \left(-\frac{4}{9}, -\frac{11}{9}\right)$, $B = \left(\frac{4}{15}, \frac{1}{5}\right)$, $C = \left(\frac{4}{7}, -\frac{5}{7}\right)$.

The vectors $\overrightarrow{AB} = \vec{B} - \vec{A} = \left(\frac{32}{45}, \frac{64}{45}\right)$ and $\overrightarrow{AC} = \vec{C} - \vec{A} = \left(\frac{64}{63}, \frac{32}{63}\right)$ M1A1

giving the area of the triangle as $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{pmatrix} \frac{32}{45} \\ \frac{64}{45} \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{64}{63} \\ \frac{32}{63} \\ 0 \end{pmatrix}\right| = \frac{512}{945} \approx 0.542$. M1A1

**13** The perpendicular bisectors of chords AB and BC (or similar) will intersect at the centre of the circle. R1

Midpoint $AB$ = $\left(\frac{1+3}{2}, \frac{2+1}{2}\right) = (2, 1.5)$ A1

gradient AB = $\frac{1-2}{3-1} = -0.5$ so gradient of perpendicular bisector of AB is $2$.

Equation of perpendicular bisector of $AB$ is $y - 1.5 = 2(x - 2) \Rightarrow y = -2.5 + 2x$. M1A1

Similarly, the perpendicular bisector of $BC$ is $y - 3 = -0.5(x - 4) \Rightarrow y = 5 - 0.5x$. A1

The bisectors will intersect at the centre of the circle, so $-2.5 + 2x = 5 - 0.5x \Rightarrow x = 3$

and $y = 3.5$. A1

Then the radius of the circle is $\sqrt{(3-1)^2 + (3.5-2)^2} = 2.5$ M1A1

**14** Using GDC,

Median $= 60$ A1

Mean $= 56.93$ A1

SD $= 20.31$ M1A1

$\frac{3(\text{mean} - \text{median})}{\text{standard deviation}} = \frac{3(56.93 - 60)}{20.31} = -0.45$ M1A1

This indicates a negative skewness, but only by a small amount. R1

The data is therefore generally symmetric about the mean. R1

# 4 Modelling constant rates of change: linear functions and regressions

**1** A skydiver falls 134.5 m in the first 5 seconds and 354.5 m in the next 5 seconds. Assume that the distance (in m) the skydiver falls each second is an arithmetic sequence with first term $a$ and common difference $d$.

**a** Find formulas in terms of $a$ and $d$ that represent the total distance fallen by the skydiver in

**i** the first five seconds **ii** the first ten seconds.

**b** Hence, find $a$ and $d$.

**c** The skydiver should open his parachute by the time he has travelled 2 km. Determine the latest second that the skydiver can open his parachute.

**2** The data in the table is collected from IB students on the average daily amount of time spent on social media (in hours) and average daily hours of sleep.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Hours of social media | 5.7 | 4.3 | 8.8 | 2.4 | 4.6 | 5.9 | 4.3 | 3.7 | 5.5 | 6.1 | 5.9 | 3.7 | 3.1 | 5.1 | 5.2 | 4.0 | 6.5 | 3.4 | 4.8 |
| Hours of sleep | 6.2 | 7.1 | 8.3 | 7.8 | 7.0 | 6.7 | 7.5 | 7.3 | 6.8 | 6.2 | 5.3 | 7.5 | 7.2 | 7.4 | 6.9 | 6.8 | 6.1 | 7.1 | 6.6 |

**a** **i** Give a reason why a linear regression may be appropriate to model this data.

**ii** Find and comment on the correlation coefficient.

**b** Create a box-and-whisker plot for each variable, including finding and removing outliers.

**c** Exclude any data points that contain an outlier in either variable and recalculate the correlation coefficient. Comment on how your result compares to the coefficient in part **a**.

**3** Pressure is defined as the amount of force (measured in newtons, N) per unit area (measured in m$^2$): $P = \frac{F}{A}$. Its units are pascals (Pa). The pressure that a submerged object experiences is a linear function of its depth: $P(d) = rgd + P_{\text{atm}}$. Here, $r$ is the density of the fluid in kg/m$^3$, $g = 9.8$ m/s$^2$ is the acceleration due to gravity, $d$ is the depth of the object below the surface of the fluid and $P_{\text{atm}} = 1.01 \times 10^5$ Pa is normal atmospheric pressure.

Seawater has a density of $1.03 \times 10^3$ kg/m$^3$.

**a** Find $P(d) = md + c$ for an object submerged $d$ m in seawater. Express the parameters in your function in the form $a \times 10^n$, where $1 \le a < 10$ and $n \in \mathbb{Z}$.

**b** Use your function to find:

**i** the pressure experienced by a diver submerged to a depth of 12 m in the ocean

**ii** the maximum safe depth to descend to if the diver should not exceed a pressure of $5.25 \times 10^5$ Pa.

**Exam-style questions**

**4** Let $f(x) = 3x$ and $g(x) = x - 4$ , both with domain of $\mathbb{R}$. Find

**a** $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $f^{-1}(x)$ **d** $g^{-1}(x)$

**e** $(f \circ g)^{-1}(x)$ **f** $(g \circ f)^{-1}(x)$ **g** $(f^{-1} \circ g^{-1})(x)$

**h** $(g^{-1} \circ f^{-1})(x)$ **e** $(f \circ f)(x)$ **f** $(f \circ f^{-1} \circ f)(x)$ (13)

**5** **a** State which of the following expressions are functions with domain $\mathbb{R}$ . For expressions which are not functions on $\mathbb{R}$ , give a reason why not.

**i** $y = \pi x + 1$ **ii** $x^2 + y^2 = 25$ **iii** $y = \sin x$

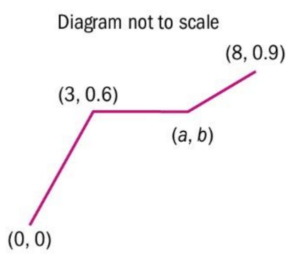
**iv** $3x + 4y + 7 = 0$ **v** $(y - 7) = 5(x + 9)$ **vi** $y = \sqrt{x} + 1$

**vii** $y = 8$ **viii** $x = 7$ **ix** $y = \dfrac{1}{x}$

**x** $y = x^2$ . (10)

**b** For the expressions in part **a** that are functions, state if they represent a straight line or not. (6)

**6** In a skateboard park the form of one of the sides is as shown in the diagram below. There is a horizontal part in the middle and the units are in metres.




The formula for this piecewise linear function is given by

$$f(x) = l \begin{cases} mx & 0 \le x \le k \\ & k \le x \le 5 \\ px + q & 5 \le x \le r \end{cases}$$

Find the values of *m, l, k, a, b, p, q, r.* (11)

**7** The data below represents the number of crimes committed and the number of policemen employed in a city during various years in the 20th century. The crimes are in units of thousands, and the policemen are in units of hundreds.

| Year | 1900 | 1910 | 1920 | 1930 | 1940 | 1950 | 1960 | 1970 | 1980 | 1990 | 2000 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Crimes, $c$ | 15 | 16 | 16 | 18 | 19 | 24 | 30 | 35 | 40 | 46 | 59 |
| Policemen, $p$ | 7 | 7 | 8 | 9 | 10 | 12 | 14 | 14 | 16 | 18 | 20 |

**a** Find the linear regression line of best fit $c$ on $p$ . (3)

**b** Find the Pearson product moment correlation coefficient and comment on the validity of the linear model found in part (a) in the light of this value. (3)

**c** In the year 1995 there were 1900 policemen employed. Estimate the number of crimes committed in that year. (2)

**d** In the year 1850 there were 50 policemen employed. Explain why the model obtained would not be used to estimate the number of crimes committed in that year. (1)

**8** A stadium has 50 rows of seating. The number of seats on row $n$ is $u_n$. The sequence $\{u_n\}$ is arithmetic, with first term $u_1 = a$ and common difference of $d$. The sum of the number of seats in the first $n$ rows of the stadium is given by $S_n = 95n + 5n^2$ .

**a** Find the total number of seats in the stadium. (2)

**b** Find the value of $a$. (2)

**c** Find $S_2$ and hence find the value of $d$ . (3)

**d** Hence write down a formula for $u_n$. (1)

The seats are numbered 1, 2, 3... along the first row and then continuing along the 2nd row, then the 3rd ... until the end of the 50th row.

**e** John has seat 15 000. Determine which row John sits in. (2)

**Answers**

**1** **a** **i** $S_5 = 2.5(2a + 4d) = 134.5$ **ii** $S_{10} = 5(2a + 9d) = 489$

**b** $a = 9.3$ m, $b = 8.8$ m

**c** $S_n = 2000$ has solution $n = 20.8$, so the skydiver should open his parachute by the 20th second.

**2** **a** **i** A scatter plot shows a negative linear correlation, although there are two apparent outliers.

**ii** $r = -0.203$, which indicates a weak negative correlation between hours spent on social media and hours of sleep.

**c** $r = -0.809$. The correlation coefficient now shows a strong negative correlation. This shows that excluding outliers can substantially increase the strength of a correlation.

**3** **a** $P(d) = 1.01 \times 10^4 d + 1.01 \times 10^5$

**b** **i** $2.22 \times 10^5$ Pa **ii** $d = 42.0$ m

**4** **a** $3(x-4) = 3x - 12$ A1 **b** $3x - 4$ A1

**c** $\frac{1}{3}x$ M1A1 **d** $x + 4$ A1

**e** $\frac{x+12}{3} = \frac{x}{3} + 4$ M1A1 **f** $\frac{x+4}{3}$ A1

**g** $\frac{x+4}{3}$ A1 **h** $\frac{x}{3} + 4$ A1

**i** $9x$ A1 **j** $3x$ M1A1

**5** **a** **i** function **ii** not a function, e.g no $y$ value if $x = 6$ or e.g. two $y$ values if $x = 3$

**iii** function **iv** function **v** function

**vi** not a function, no $y$ value if $x < 0$ **vii** function

**viii** not a function, e.g no $y$ value if $x = 6$ or e.g. many $y$ values if $x = 7$

**ix** not a function, no $y$ value if $x = 0$ **x** function

A1A1A1A1A1A1A1A1A1A1

**b** **i** straight line **iii** not a straight line **iv** straight line

**v** straight line **vii** straight line **x** not a straight line

A1A1A1A1A1A1

**6** Gradient $m = \frac{0.6}{3} = 0.2$ M1A1

$l = 0.6$ A1

$k = 3$ A1

$a = 5$ A1

$b = 0.6$ A1

Gradient $p = \frac{0.9 - 0.6}{8 - 5} = 0.1$ M1A1

$0.6 = 0.1 \times 5 + q \Rightarrow q = 0.1$ M1A1

$r = 8$ A1

**7** **a** $c = 3.17p - 9.96\ (3sf)$ M1A1A1

**b** $r = 0.976(3sf)$ showing strong positive linear correlation, so a linear model is valid. M1A1R1

**c** $3.1668... \times 19 - 9.956... = 50.2$ so $50\,200(3sf)$ crimes. M1A1

**d** Not valid as this is extrapolation. R1

**8** **a** $95 \times 50 + 5 \times 50^2 = 17250$ M1A1

**b** $a = u_1 = S_1 = 100$ M1A1

**c** $S_2 = 95 \times 2 + 5 \times 2^2 = 210$ A1

$u_2 = 110 \Rightarrow d = 10$ M1A1

**d** $u_n = 100 + (n-1)5 = 5n + 95$ A1

**e** Need the smallest value of $n$ such that $S_n \geq 15\,000$ R1

John sits in row 47 A1

# 5 Quantifying uncertainty: probability

**1** In Alex's class, 91% of the students passed physics and 85% passed chemistry. 80% of the students passed both physics and chemistry. A student is chosen at random from the class.

**a** Determine if the events "the student passed physics" and "the student passed chemistry" are independent.

**b** A randomly chosen student passed physics. Find the probability that he/she did not pass chemistry.

**2** Three numbers are chosen at random from the set {1,2,3,4}.

**a** Find the probability that the three numbers can form consecutive terms of an arithmetic sequence.

**b** Find the probability that the three numbers can form consecutive terms of a geometric sequence.

**3** The coffee machine in the Pi café has three coffee dispensers P, Q and R. The probabilities that each malfunctions on a given day are 0.007, 0.003 and 0.001 respectively.

**a** Find the probability that exactly one of the dispensers malfunctions on a given day.

**b** Find the probability that at least one of the dispensers will malfunction on a given day.

**4** In a maths competition, a team of four mathematicians have ten minutes to tackle a tough problem. Student A has 10% chance of solving the problem. Student B has 11%, student C 17% and student D 8%. If they all try the problem independently of each other, find the probability that after the ten minutes, the group will have solved the problem.

**5** A student is chosen at random from an IB class. *C* is the event "a student takes chemistry", and *P* is the event "a student takes physics". You are given that
$P(C) = 0.4, P(P|C) = 0.6, P(P|C') = 0.5$

**a** Calculate the probability that a student chosen at random studies physics.

**b** Given that a randomly chosen student studies physics, find the probability that the student studies chemistry.

**6** Two rugby players Achille and Barbora kick for goal at the same time. If Barbora is three times as likely to score a goal as Achille, and if the probability of a goal being scored is ½, find the probability that Barbora scores a goal.

**7** An integer is chosen at random from the first 2000 positive integers. Find the probability that the integer is

**a** A multiple of 6 **b** A multiple 8 **c** A multiple of *both* 6 and 8.

**d** Determine if the events "choose a multiple of 6" and "choose a multiple of 8" are independent.

**8** A health survey in a large group of college students reveals that 70% use headphones frequently to listen to their MP3 player while the rest used external speakers or used headphones infrequently. In total, 40% of the students surveyed had significant hearing loss, 20% had moderate hearing loss and the rest had no hearing loss.

Of the students using headphones frequently, 50% were found to have significant hearing loss and 20% had moderate hearing loss.

A student is chosen at random from the students surveyed.

**a** Determine if the events "the student uses headphones frequently" and "the student has severe hearing loss" are independent.

**b** If the findings of the survey are replicated in a college of 12,502 students, predict the expected number of students who have no hearing loss?

**Exam-style questions**

**9** Albert and Eric play a game involving a biased dice, where $P(\text{throwing a } 6) = \frac{1}{5}$

The first player to throw a 6 wins the game. Albert throws first, then Eric.

**a** Find the probability that Eric wins (5)

**b** The players play the game 50 times.
How many times can Albert expect to win? (3)

**10** The probability that Steve the striker scores in any given football match is 0.28.

Four matches over the course of a season are selected at random.

**a** Find the probability that Steve scores in at least one of the four matches. (3)

**b** Find the probability that Steve scores in exactly one of the four matches. (2)

**c** Find the probability that Steve scores in at least two of the four matches. (3)

**11** A survey of Welsh sports fans was taken and was is found that everyone either supported their local football team, or their local rugby team, or both.

The probability that a randomly selected person supports both their rugby and football teams is $x$. The probability that someone supports their football team, given that they support their rugby team, is $\frac{1}{5}$. The probability that someone supports their rugby team, given that they support their football team, is $\frac{1}{6}$. Find the probability that a randomly selected person supports only one sport. (8)

**12** A bag contains $x$ red balls and $3$ blue balls.
Amber takes out one ball at random, then selects and takes another.
The probability that Amber has taken out two balls of different colours is $\frac{9}{22}$.

**a** Show that $3x^2 - 29x + 18 = 0$. (6)

**b** Hence find the value of $x$. (3)

**Answers**

**1** **a** The events are not independent.

**b** 0.11

**2** **a** 2/5 **b** 1/10

**3** **a** 0.010938 **b** 0.010969

**4** 0.388356

**5** **a** 0.54 **b** 4/9

**6** 0.418861

**7** **a** 0.166 **b** 0.125 **c** 0.041

**d** they are not independent.

**8** **a** they are not independent

**b** 5001.

**9** **a** $P(\text{Eric wins}) = \left(\frac{4}{5}\times\frac{1}{5}\right)+\left(\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{1}{5}\right)+\ldots$ M1A1

Use of infinite geometric series formula M1

$$=\frac{\left(\frac{4}{5}\times\frac{1}{5}\right)}{1-\left(\frac{16}{25}\right)}$$ A1

$$=\frac{4}{9}(=0.44)$$ A1

**b** $P(\text{Albert wins}) = 1-\frac{4}{9}=\frac{5}{9}$ M1A1

$\frac{5}{9}\times 50 = 28$ times A1

**10** **a** $1-P(\text{no scoring}) = 1-0.72^4$ M1A1

$=0.731$ A1

**b** $4\times 0.28\times 0.72^3$ M1

$=0.418$ A1

**c** $1-P(\text{no goals})-P(\text{exactly one goal})$ M1

$=1-0.72^4-0.418$ A1

$=0.313$ A1

**11** $P(F\cap R)=P(F|R)P(R)$ M1

So $x=\frac{P(R)}{5}$ and $P(R)=5x$ A1

$P(R\cap F)=P(R|F)P(F)$ M1

So $x=\frac{P(F)}{6}$ and $P(F)=6x$ A1

P(football only) $= 6x - x = 5x$ A1

P(rugby only) $= 5x - x = 4x$ A1

So P(one sport) $= \dfrac{5x + 4x}{10x}$ M1

$= \dfrac{9}{10}$ A1

**12 a** $\dfrac{x}{x+3}\left(\dfrac{3}{x+2}\right) + \dfrac{3}{x+3}\left(\dfrac{x}{x+2}\right) = \dfrac{9}{22}$ M1A1A1

$\dfrac{6x}{(x+3)(x+2)} = \dfrac{9}{22}$ A1

$\dfrac{2x}{(x+3)(x+2)} = \dfrac{3}{22}$

$44x = 3(x+3)(x+2)$ A1

$44x = 3x^2 + 15x + 18$ A1

$3x^2 - 29x + 18 = 0$ AG

**b** $(3x-2)(x-9) = 0$ M1A1

$\therefore x = 9$ , (since $x \in \mathbb{N}$ ) A1

# 6 Modelling relationships with functions: power and polynomial functions

**1** The first 4 terms of an arithmetic sequence are 3, 7, 11, 15

**a** Find an expression for the sum of the first n terms.

**b** If the sum of the first n terms is equal to 741, sketch the graph and find the value for n.

**2** A cone with radius, r, and perpendicular height, h=2r.

**a** Write down an equation for the volume, V, in terms of r and h.

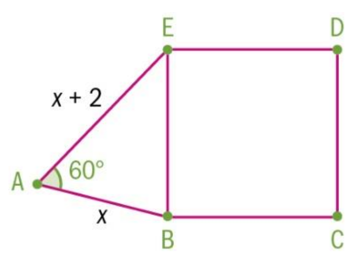
**b** Find an expression for the total surface area, A, of the cone, writing your answer in terms of r.

**3** A right triangle has sides x and (10-x).

**a** Determine the area of the triangle.

**b** Find the maximum possible area of the triangle.

**4** In the following shape




**a** Determine side BE in terms of x.

**b** Hence find an expression for the area of the square BCDE in terms of x.

**c** Find the range of values that the area of square BCDE can take.

**5** In a class of 10 students, x have studied for an upcoming test, where x>3. Three students from the class are selected at random.

Determine an expression for the probability that all three students have studied.

**6** Anmol throws a stone in the air. The height of the stone, h(t) metres, at time t seconds is modelled by the equation $h(t) = -2.2625x^2 + 8.575x + 1.9$.

**a** Find the y – intercept and explain what this represents.

**b** Find the maximum height of the stone.

**c** Find the rate of change in metres per second between x = 1.8 and x = 1.9 and decide if the rate of change is increasing or decreasing.

**d** Find the time when the stone lands back on the ground.

**7** The number of miles, m, travelled varies directly as the speed, s, of a train.

**a** If the train travels 100 km in 1.25 hours, write an equation connecting m and s.

**b** Find the number of miles travelled after 2 hours.

**c** Find how long it takes to travel 300 km.

**8** The line $2y = x + 3$ intersects the curve $f(x) = x^2 + 2x - 5$ at the points A and B. Find the coordinates of A and B.

**9** Consider the continuous piecewise function $f(x)$, defined as:

$$f(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ c - mx \quad \text{for} & 2 < x \le 6 \\ 0 & \textit{otherwise} \end{cases}.$$

**a** Determine the values of the constants c and m.

**b** Hence sketch the graph of the function.

**10** Consider the continuous piecewise function $f(x)$, defined as:

$$f(x) = \begin{cases} x^2 + a & 0 \le x \le 10 \\ \dfrac{11000}{x^2} \quad \text{for} & 10 < x \le 20 \\ 0 & \textit{otherwise} \end{cases}.$$

**a** Determine the value of constant a.

**b** Hence sketch the graph of the function.

**c** Determine the value(s) of x for which $f(x) = 60$

**11** A skydiver is about to jump from a plane that is 1000 m above the ground. At first while he falls freely under gravity, his height above the ground is given by the function $h(t) = a - 5t^2$, where t is the time (in secondas) after he jumps from the plane, h is his height above the ground (in meters) and a is a positive constant. After 10 seconds, he opens his parachute, which slows his decent and from that point on his height above the ground is given by the function $h(t) = b - 10t$, where *b* is a positive constant.

**a** Find the value of the positive constant *a*.

**b** Determine the height of the skydiver above the ground at the moment he opens his parachute.

**c** Hence find the value of the constant *b*.

**d** Determine how long it will take for the skydiver to reach the ground.

**e** Write the function modelling his fall in piecewise function notation.

**f** Sketch the graph of $h(t)$ in an appropriate domain.

**Exam-style questions**

**12** The quadratic equation $x^2 + 7x + 2 = 0$ has two roots $\alpha$ and $\beta$. In this question, you do not need to find $\alpha$ and $\beta$.

**a** Write down the value of **i** $\alpha + \beta$ **ii** $\alpha\beta$ (2)

**b** Find the value of **i** $(\alpha+\beta)^2$ **ii** $\alpha^2+\beta^2$ **iii** $(\alpha-\beta)^2$

**iv** $\alpha^3+\beta^3$ (8)

**c** Another quadratic equation $y^2+py+q=0$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. Find the values of $p$ and $q$. (4)

**13** A function is defined by $f(x)=\sqrt{x+4}-3$. It is defined on its largest possible domain.

**a** Write down this largest possible domain. (1)

**b** For this domain, write down the range of the function. (1)

**c** Find the inverse function $f^{-1}(x)$. (3)

**d** For the inverse function state (i) its domain (ii) its range. (2)

**e** Solve $f^{-1}(x)=0$ (3)

**14** If a factory produces $n$ cars in a day, it makes a profit of $\$P$ given by $P=n^2-20n$.

Find how many cars the factory must produce to

**a** make a profit (3)

**b** make a profit of $4800 (2)

**c** make the worst possible loss, and state the value of this loss. (3)

**15** Consider the equation $(k-2)x^2+2x-k=0$.

**a** Show that this equation always has real roots whatever the value of the constant $k$ is. (6)

**b** State the values of $k$ for which there is only one real root and state the value of this one root. (6)

**16** Two ice figure skaters leave tracks on the ice. These tracks are given by the curves

$y=x^2+\frac{1}{8}$ and $y=3x-x^2-1$. Show that these curves do not cross but just touch at one point. Find the coordinates of this common point. (6)

**Answers**

**1** **a** $n+2n^2$

**b** 19

**2** **a** $\frac{1}{3}\pi^2 rh$

**b** $\pi r^2\left(1+\sqrt{5}\right)$

**3** **a** $\frac{1}{2}(10-x)x$

**b** 12.5

**4** **a** $\sqrt{4+2x+x^2}$

**b** $4+2x+x^2$

**c** $4+2x+x^2 \geq 3$

**5** $\dfrac{x}{10}\times\dfrac{x-1}{9}\times\dfrac{x-2}{8}$

**6** **a** $y=1.9$, the initial height of the throw

**b** 10.025 m

**c** 0.43 at $x=1.8$ and -0.0225 at $x=1.9$, so decreasing

**d** 4s

**7** **a** $m=80s$

**b** 160 km = 99.4 miles

**c** 3.75 h

**8** $1.91, -3.41$

**9** **a** $c=6, m=1$

**10** **a** $a=10$

**c** $7.07, 13.54$

**11** **a** 1000

**b** 500

**c** 600

**d** 60s

**e** $h(t)=\begin{cases}1000-5t^2, & 0\leq t\leq 10\\ 600-10t, & 10<t\leq 60\end{cases}$

**12** **a** **i** $-7$ **ii** 2 A1A1

**b** **i** 49 **ii** $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=49-4=45$ A1M1A1

**iii** $(\alpha-\beta)^2=\alpha^2+\beta^2-2\alpha\beta=45-4=41$ M1A1

**iv** $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha^2\beta-3\alpha\beta^2=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$ M1M1

$=(-7)^3-3\times2\times(-7)=-301$ A1

**c** Sum of roots $=-p=\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}=\frac{-7}{2}\Rightarrow p=\frac{7}{2}$ M1A1

Product of roots $=q=\frac{1}{\alpha}\times\frac{1}{\beta}=\frac{1}{\alpha\beta}=\frac{1}{2}$ M1A1

**13 a** $x\geq-4$ A1

**b** $y\geq-3$ A1

**c** $y=\sqrt{x+4}-3$ will mean that inverse is given by
$x=\sqrt{y+4}-3\Rightarrow x+3=\sqrt{y+4}\Rightarrow y=(x+3)^2-4$ M1A1

So $f^{-1}(x)=(x+3)^2-4$ A1

**d** **i** $x\geq-3$ **ii** $y\geq-4$ A1A1

**e** $(x+3)^2-4=0\Rightarrow x+3=\pm2\Rightarrow x=-5\text{ or }-1$ M1A1

but since domain is $x\geq-3$, so $x=-1$ R1

**14 a** $n^2-20n=0\Rightarrow n(n-20)=0$ with roots of 0 and 20. So to make a profit need $n\geq21$
M1A1A1

**b** $n^2-20n=4800\Rightarrow n=80$ M1A1

**c** Min of quadratic is at $n=\frac{20}{2\times1}=10$ Loss is $100-20\times10=-\$100$ M1A1A1

**15 a** If $k\neq2$ the equation is a quadratic

Discriminant $=2^2-4(k-2)(-k)=4+4k^2-8k=4(k^2-2k+1)=4(k-1)^2$ M1A1A1

This is always $\geq0$ so there are real roots R1AG

Special case when $k=2$, equation is $2x-2=0$ which has one real root of $x=1$ R1

**b** In the special case when $k=2$, there is only one root $x=1$ A1

In the case when $k\neq2$, Quadratic has one root $\Rightarrow4(k-1)^2=0\Rightarrow k=1$ R1M1A1

Equation is $-x^2+2x-1=0\Rightarrow-(x-1)^2=0\Rightarrow x=1$ M1A1

**16** $x^2+\frac{1}{8}.=3x-x^2-1.\Rightarrow2x^2-3x+\frac{9}{8}=0\Rightarrow2(x-\frac{3}{4})^2=0$ M1A1A1

Only one solution so the curves do not cross but just touch R1AG

at $x=\frac{3}{4},y=\frac{11}{16}$ A1A1

# 7 Modelling rates of change: exponential and logarithmic functions

**1** The second, eleventh and seventeenth terms of a not constant arithmetic progression are equal to the second, third and fourth terms of a not constant geometric progression.

**a** Find the common ratio of the geometric progression.

**b** The first term of the arithmetic progression is 84. Find the common difference.

**c** Find the sum of the infinite geometric series.

**2** Consider the function $f(x) = \frac{1}{1+e^{-x}}$, where $x \in \Re$.

**a** Determine the equations of the two asymptotes of $f(x)$.

**b** Given that $f(x)$ is an increasing function, find its range.

**c** Find the inverse function $f^{-1}(x)$.

**Exam-style questions**

**3** David buys a car for \$20 000 that depreciates at 20% per year.

At the same time, Laura buys a car for \$2000 that depreciates at 3% per year.

In this question, give all monetary answers to the nearest dollar (\$) and all values of time in years correct to one decimal place.

**a** After two years, find the value of **i** David's car **ii** Laura's car. (3)

**b** Find how many years it takes for David's car to be worth \$5000. (3)

**c** Find how many years it takes for both cars to have the same value, and state what this value is. (4)

**4** In this question, give all answers in an exact form.

**a** Solve $(\log x)^2 - \log x - 12 = 0$. (4)

**b** Solve $e^{2x} - e^{x} - 30 = 0$. (5)

**5** The length $x$ of a narwhal's tusk is given by $x = \frac{2}{1+19e^{-ct}}$ where $x$ is measured in metres and $t$ is time in years since its birth. The constant $c$ is positive.

**a** Find the length $x$ at birth. (1)

**b** As the narwhal becomes extremely old, state what value $x$ tend towards, and justify your answer. (2)

You are given the extra information that when $t = 4$, then $x = 1$.

**c** Find the value of the constant $c$

**i** in an exact form

**ii** correct to 3 significant figures. (4)

**d** If $t = 8$ find the value of $x$. (2)

**e** If $x = 1.99$ find the value of $t$. (2)

**6** Let $\log_a b = x$. This is equivalent to the statement $b = a^x$ .

**a** By taking logarithms to base $b$ in the equation $b = a^x$, prove that $\log_a b = \dfrac{1}{\log_b a}$. (3)

**b** Hence, solve the equation $\log x + 2 - 15\log_x 10 = 0$ (6)

**7** A sequence that is mixture of arithmetic and geometric is defined by $u_n = (a + (n-1)d)r^{n-1}$ where $r \neq 1$ . The sum of the first $n$ terms of this sequence is given by

$$S_n = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots + (a+(n-1)d)r^{n-1} .$$

**a** Take the above equation for $S_n$. Multiply both sides by $r$, and then write it underneath the original equation, but with the terms displaced to the right by one term (so that terms with the same power of $r$ are aligned vertically).

Use this to find and simplify an expression for $(1-r)S_n$ .

Hence, prove that $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{(a+(n-1)d)r^n}{1-r}$ (8)

Julia is going to invest in a bank offering 5% compound interest per year.

On the 1st January of year 1 she pays in £100. On the 1st January of year 2 she pays in £200. On 1st January of year 3 she pays in £300 and so on, until the last instalment of £1000 on the 1st January in year 10.

**b** Calculate how much money Julia pays into the bank in total. (2)

**c** Use part **b** to find the total value of her investment on 1st January of year 10 just after she has deposited the last sum of £1000, giving the answer to 2 decimal places. (6)

**Answers**

**1** **a** $\frac{2}{3}$

**b** -3

**c** 364.864

**2** **a** $f(x)=0, f(x)=1$

**b** $0<f(x)<1$

**c** $\log\frac{y}{1-y}$

**3** **a** **i** $20000\times(0.8)^2=\$12800$ **ii** $2000\times(0.97)^2=\$1882$ M1A1A1

**b** Solving $20000\times(0.8)^x=\$5000 \Rightarrow x=6.21\ldots$ so $6.2$ years M1A2

**c** $20000\times(0.8)^x=2000\times(0.97)^x \Rightarrow x=11.95\ldots$ so 12.0 years M1A1

At this time, cars are worth $2000\times 0.97^{11.95004\ldots}=\$1390$ M1A1

**4** **a** $(\log x-4)(\log x+3)=0 \Rightarrow \log x=4 \text{ or } -3 \Rightarrow x=10^4 \text{ or } 10^{-3}$ M1A1A1A1

**b** $(e^x)^2-e^x-30=0 \Rightarrow (e^x-6)(e^x+5)=0 \Rightarrow e^x=6 \text{ or } -5$ M1M1A1

$e^x=-5$ is impossible if $x$ real, so $x=\ln 6$ R1A1

**5** **a** $\frac{2}{1+19}=0.1m$ A1

**b** $\lim_{t\to\infty} e^{-ct}=0 \Rightarrow x\to 2$ m as the Narwhal becomes old R1A1

**c** $1=\frac{2}{1+19e^{-4c}} \Rightarrow 1+19e^{-4c}=2 \Rightarrow e^{-4c}=\frac{1}{19} \Rightarrow -4c=\ln\frac{1}{19}$ M1A1

**i** $c=\frac{\ln 19}{4}$ **ii** 0.736 A1A1

**d** $x=\frac{2}{1+19e^{-t\frac{\ln 19}{4}}}$, when $t=8 \Rightarrow x=1.90$ m(3 s.f.) A2

**e** $x=1.99 \Rightarrow t=11.2$ years(3 s.f.) A2

**6** **a** $\log_b b=\log_b a^x \Rightarrow 1=x\log_b a \Rightarrow \frac{1}{\log_b a}=x=\log_a b$ M1A1M1AG

**b** $-15\log_x 10=\frac{-15}{\log x}$ by part **a** M1

$\log x+2-\frac{15}{\log x}=0 \Rightarrow (\log x)^2+2\log x-15=0 \Rightarrow (\log x-3)(\log x+5)=0$ M1A1

$\log x=3 \text{ or } -5$ so $x=10^3$ or $10^{-5}$ A1A1A1

**7** **a** $S_n = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots + (a+(n-1)d)r^{n-1}$

$rS_n = ar + (a+d)r^2 + (a+2d)r^3 + (a+3d)r^4 + \ldots + (a+(n-1)d)r^n$ A1

Subtracting

$(1-r)S_n = \ a + \ dr + dr^2 + dr^3 + dr^4 + \ldots + dr^{n-1} - (a+(n-1)d)r^n$ M1A1

Apart from the 1st and last terms, this is a GP with $n-1$ terms, 1st term of $dr$ and common ratio of $r$ so R1

$(1-r)S_n = \ a + \ \dfrac{dr(1-r^{n-1})}{(1-r)} - (a+(n-1)d)r^n$ . M1A1A1

Giving $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{(a+(n-1)d)r^n}{1-r}$ AG

**b** $100 + 200 + 300 + \ldots + 1000 = \dfrac{10}{2}(100+1000) = £5500$ M1A1

**c** 10th instalment is worth $1000$, 9th instalment is worth $900 \times 1.05$, 8th instalment is worth $800 \times (1.05)^2 \ldots$ R1

So we can use part **a** with $a = 1000,\ d = -100, r = 1.05,\ n = 10$ R1A2

$\dfrac{1000}{-0.05} + \dfrac{-100 \times 1.05(1-1.05^9)}{0.05^2} - \dfrac{100 \times 1.05^{10}}{-0.05} = £6413.57$ M1A2

# 8 Modelling periodic phenomena: trigonometric functions and complex numbers

**1** A map of Australia is shown with the international airports marked in purple. A Voronoi diagram has been drawn using these points.


**a** An aircraft is flying over Uluru (Ayers Rock) and is experiencing engine trouble so needs to travel to the nearest international airport. Where should they go?

**b** Canberra has a smaller airport. To fly from there which would be the nearest international airport?

**c** To travel to Tasmania, which international airport is nearest?

**d** If Perth airport were not operating, which would be the nearest airport for a plane directly over Perth? Explain your reasoning.

Due to the large distances over part of Australia, the government proposes building a runway for emergency use.

**e** Discuss where would be the best place to build the runway.

**2** If $A(2, \ -1, \ 1)$, $B(0, \ 1, \ 2)$, and $C(1, \ 1, \ 3)$, find:

**a** the coordinates of the point $D$ such that $ABCD$ is a parallelogram

**b** the angle $ABC$

**c** $\overrightarrow{BA} \times \overrightarrow{BC}$

**d** the area of *ABCD*

**e** the equation of the line passing through *B* and *D*

**3** The sixth term of a geometric sequence is 96 and the eleventh term is 3. Find the first term and the common ratio. Show that the series converges, and find the sum to infinity.

**4** A psychologist wants to investigate the relationship between the IQ of a child and the IQ of the mother. She measures the IQ of a sample of eight children and their mothers.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Child's IQ ($x$) | 87 | 91 | 94 | 98 | 103 | 108 | 111 | 123 |
| Mother's IQ ($y$) | 94 | 96 | 89 | 102 | 98 | 94 | 116 | 117 |

**a** Find the correlation coefficient between the child's IQ and the mother's IQ.

**b** The psychologist wishes to estimate the IQ of a mother of a child with IQ 100.

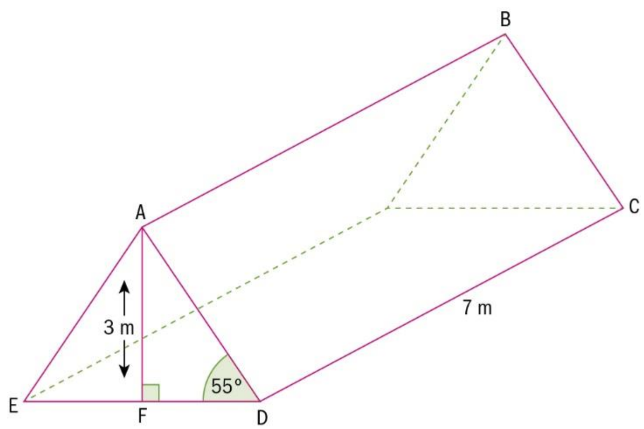
**i** If the $y$-on-$x$ regression line is used to calculate this estimate, state which of the variables would need to correspond to $x$.

**ii** Write down the equation of the regression line used

**iii** Write down the estimate obtained.

**c** If a mother has an IQ of 110, estimate the IQ of her child, showing your working clearly.

**5** The following diagram shows a right isosceles triangular prism. The angle ADE is 55°. The vertical height, AF, of the prism is 3 m and the length DC is 7 m.




**a** Calculate AD.

**b** Calculate the length of the diagonal DB.

**c** Calculate the surface area of the prism

**6** Carlo saves \$100 at the beginning of every month for a period of 50 months. The money is saved in an account that pays 6% annual interest calculated monthly.

**a** Show that the first amount of \$100 deposited becomes \$128.32 by the end of Carlo's saving period (the end of the 50th month).

**b** Show that the second amount deposited becomes \$127.68 by the end of Carlo's saving period.

**c** Express the amounts saved and the interest received as a series and hence find the total amount Carlo has saved at the end of the 50 months.

**Exam-style questions**

**7** A duck is floating on the surface of the lake. Its position above the bottom of the lake can be modelled by the function $D(t) = 100 + A\cos(\phi + \omega t)$ m, where $t \geq 0$ and $\phi$ is a constant.

**a** When $t = 0$ s, the duck is at its highest point of oscillation. Find the constant $\phi$. (3)

**b** When $t = 10$ s, the duck is at its lowest point of oscillation. If in those 10 s the duck has reached the highest point 3 times, find the constant $\omega$. (4)

**c** When $t = \frac{1}{6}$ s, $D(t)$ = 105 m. Find the amplitude of the duck's motion, $A$. (2)

**8** $f(x) = \sqrt{2}\sin\left(\frac{2\pi}{3} + 5x\right)$, and $g(x) = \sqrt{3}\cos\left(\frac{\pi}{4} + 5x\right)$. Express the functions $f(x)$ and $g(x)$ in the complex forms $\text{Re}\left(re^{i\theta}\right)$. (3)

**9** Consider the function $f(x) = ax + b$, where $a$ and $b$ are constants.

The domain of $f$ is $-2 \leq x \leq 5$ and the function goes through the points $(-2, 9)$ and $\left(\frac{1}{2}, 3\right)$.

**a** Determine the exact values of $a$ and $b$. (4)

**b** Find the range of $f$. (4)

**c** Find an expression for $f^{-1}(x)$ and state its domain. (4)

**d** Sketch the graph of $y = f^{-1}(x)$. (3)

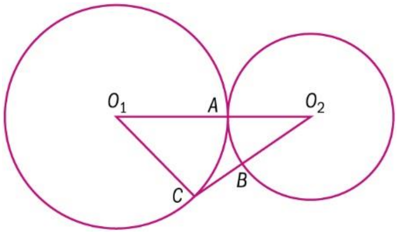
**10** Nicky invests \$10 000 at a simple interest rate of 2.1% per annum indefinitely.

At the same time, Helen invests \$9500 at a simple interest rate of 2.3% per annum.

**a** Whose savings will reach a total of \$12 000 first? (6)

**b** After how many whole years will the value of their savings to be equal? (5)

**11** The diagram shows two circles with centres $O_1$ and $O_2$, which touch at point $A$. One circle has radius $O_1A = 1$ cm, and the other has radius $O_2A = \sqrt{3} - 1$ cm. Triangle $O_1CO_2$ is such that the point $C$ lies on the circle $O_1$, and the angle $O_1\hat{C}O_2$ is $\frac{2\pi}{3}$ radians.




**a** Find the exact size of angles $O_2\hat{O}_1C$ and $O_1\hat{O}_2C$. (8)

**b** Find the exact area of the curved shape $ABC$. (8)

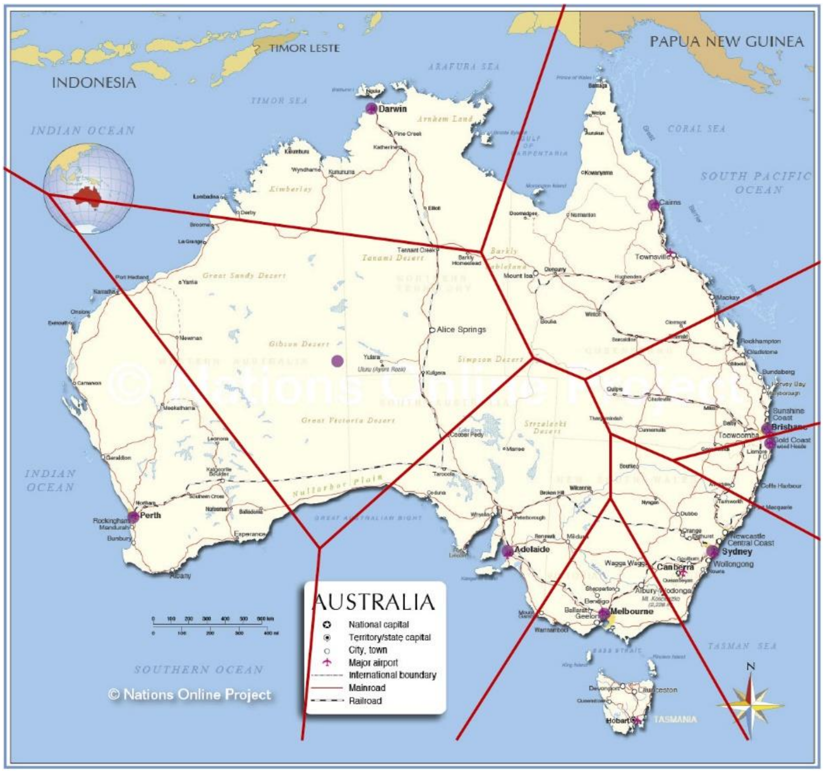
**c** Find the exact perimeter of the curved shape $ABC$. (6)

**Answers**

**1** **a** Adelaide **b** Sydney **c** Melbourne

**d** Adelaide. The perpendicular bisector between Darwin and Adelaide passes North of Perth

**e** At the intersection point to the West of Uluru or the intersection point close to Alice springs. Note, there would be an argument for moving it a little to Alice Springs so that the infrastructure is better (rail, road, emergency services etc.). Resultant Voronoi diagram shown below.




**2** **a** $\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ so coordinates are $(3 \quad -1 \quad 2)$

**b** $\overrightarrow{BA} \bullet \overrightarrow{BC} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 = 3 \times \sqrt{2} \times \cos\theta \Rightarrow \theta = 76.4^\circ$

**c** $\overrightarrow{BA} \times \overrightarrow{BC} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$

**d** $\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \sqrt{4 + 9 + 4} = \sqrt{17}$

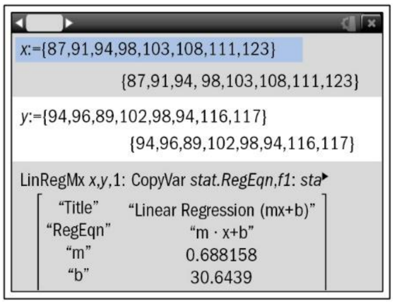
**e** $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

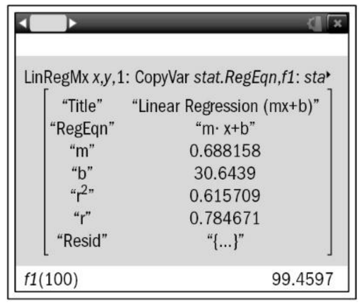
**3** $u_1 r^5 = 96$ and $u_1 r^{10} = 3$. Dividing $r^5 = \frac{3}{96} = \frac{1}{32} \Rightarrow r = \frac{1}{2}$.

$u_1 \times \frac{1}{32} = 96 \Rightarrow u_1 = 96 \times 32 = 3072$

$|r| < 1 \Rightarrow$ series converges. $S_\infty = \frac{u_1}{1-r} = 3072 \div \frac{1}{2} = 6144$

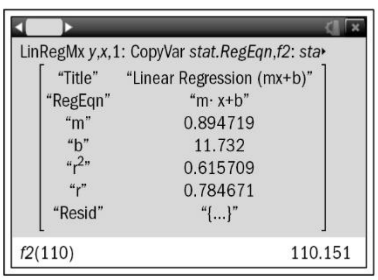
**4** **a** $r = 0.785$




**b** **i** The Child's IQ **ii** $y = 0.688x + 30.6$ **iii** 99.5

**c** Switch variables




Equation to use is $x = 0.895y + 11.7$ so estimate of child's IQ is 110

**5** **a** $\frac{3}{\text{AD}} = \sin 55^\circ \Rightarrow \text{AD} = 3.66$ m

**b** $\text{DB} = \sqrt{7^2 + \text{AD}^2} = 7.90$ m

**c** $\text{ED} = 2 \times \frac{3}{\cos 55^\circ} = 10.5$ m. $A = 3 \times \text{ED} + 7(2\text{AD} + \text{ED}) = 10\text{ED} + 14\text{AD} = 156$ m$^2$

**6** **a** $100 \times 1.005^{50} = \$128.32$

**b** $100 \times 1.005^{49} = \$127.68$

**c** $100 \times 1.005^{50} + 100 \times 1.005^{49} + 100 \times 1.005^{48} + \ldots + 100 \times 1.005^{1}$

$100 \times 1.005(1 + 1.005 + \ldots + 100 \times 1.005^{49}) = \frac{100 \times 1.005(1.005^{50} - 1)}{1.005 - 1} = \$5692.84$

**7** **a** When $t = 0$ s, $D(0) = 100 + A\cos(\phi) = 100 + A$, and hence $\phi = 0$. M1A1A1

**b** In 10 s the duck makes 2.5 full oscillations R1

Therefore it takes $\frac{10}{2.5} = 4$ s to complete an oscillation. M1

Therefore $\omega = \frac{2\pi}{4} = \frac{\pi}{2}$ rad/s. M1A1

**c** $105 = 100 + A\cos\left(\frac{\pi}{2}\times\frac{1}{6}\right) \Rightarrow A = \dfrac{5}{\cos\left(\frac{\pi}{12}\right)} = 5.18$ M1A1

**8** $\sqrt{2}\sin\left(\frac{2\pi}{3}+5x\right) = \sqrt{2}\cos\left(\frac{2\pi}{3}-\frac{\pi}{2}+5x\right) = \mathrm{Re}\left(\sqrt{2}\,e^{i\left(\frac{\pi}{6}+5x\right)}\right)$ M1A1

$\sqrt{3}\cos\left(\frac{\pi}{4}+5x\right) = \mathrm{Re}\left(\sqrt{3}\,e^{i\left(\frac{\pi}{4}+5x\right)}\right)$ A1

**9** **a** $a = \dfrac{3-9}{\frac{1}{2}-(-2)} = \dfrac{-6}{\frac{5}{2}} = -\dfrac{12}{5}$ M1A1

So $f(x) = -\dfrac{12}{5}x + b$

Substituting $\left(\frac{1}{2},3\right)$ gives $3 = -\dfrac{12}{5}\times\dfrac{1}{2}+b$. M1

So $b = \dfrac{21}{5}$ A1

$f(x) = -\dfrac{12}{5}x + \dfrac{21}{5}$

**b** $f(-2) = 9$ M1

$f(5) = -\dfrac{39}{5}$ M1

The range is therefore $-\dfrac{39}{5} \le f(x) \le 9$ A2

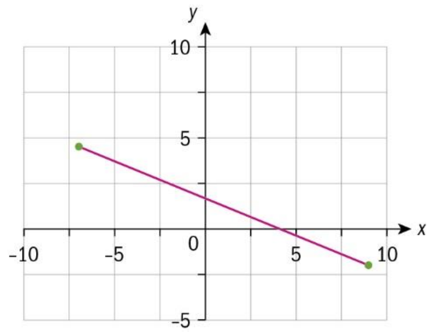
**c** $x = -\dfrac{12}{5}y + \dfrac{21}{5}$ M1

$5x = -12y + 21$ A1

$12y = 21 - 5x$

$y = \dfrac{21-5x}{12}$ A1

$f^{-1}(x) = \dfrac{21-5x}{12}$, $-\dfrac{39}{5} \le x \le 9$ AG for function; A1 for correct domain




**d**

A1 for correct line; A2 for correct endpoints of domain

**10 a** Using $I = C \times r \times n$ for Nicky: M1

$2000 = 10\,000 \times 0.021n$ A1

$\Rightarrow n = 9.52$ A1

Using $I = C \times r \times n$ for Helen:

$2500 = 9500 \times 0.023n$ M1

$\Rightarrow n = 11.44$ A1

So Nicky's savings will reach $\$12\,000$ first. A1

**b** Suppose Nicky gains $\$x$ in total interest when their savings are equal.

Then Helen must gain $\$(x + 500)$ R1

So $x = 10\,000 \times 0.021n$ A1

and $x + 500 = 9500 \times 0.023n$ A1

Solving simultaneously (using GDC) gives $n = 58.8$ M1

Equal after 59 years. A1

**11 a** Using the cosine rule $3 = 1 + (CO_2)^2 - 2 \times 1 \times (CO_2) \times \cos\frac{2\pi}{3}$ M1

$\Rightarrow (CO_2)^2 + (CO_2) - 2 = 0$ A1

Solve using GDC $\Rightarrow$ positive root is $CO_2 = 1$ cm. A1

Using the sine rule $\frac{1}{\sin O_2\hat{O}_1C} = \frac{\sqrt{3}}{\sin\frac{2\pi}{3}} \Rightarrow O_2\hat{O}_1C = \frac{\pi}{6}$. M1A1A1

Because $O_1C = O_2C$, triangle is isosceles and hence $O_1\hat{O}_2C = \frac{\pi}{6}$. R1A1

**b** The area of the triangle $O_1CO_2$ is

$\frac{1}{2}(O_1O_2) \times (O_2C) \times \sin(O_1\hat{O}_2C)$ (or equivalent, using other sides/angles) M1

Area triangle $O_1CO_2$ = $\frac{1}{2}\times\left(\sqrt{3}\right)\times(1)\times\sin\frac{\pi}{6}=\frac{\sqrt{3}}{4}$ A1

Area sector $AO_1C$ = $\frac{1}{2}\times\frac{\pi}{6}\times 1^2=\frac{\pi}{12}$ cm$^2$ and M1A1

Area sector $AO_2B$ = $\frac{1}{2}\times\frac{\pi}{6}\times(\sqrt{3}-1)^2=\frac{(\sqrt{3}-1)^2\pi}{12}=\frac{1}{6}(2-\sqrt{3})\pi$ cm$^2$ A1

Area curved shape ABC =

(Area triangle $O_1CO_2$ ) – (Area sector $AO_1C$ ) – (Area sector $AO_2B$ ) M1

$=\frac{\sqrt{3}}{4}-\frac{\pi}{12}-\frac{1}{6}(2-\sqrt{3})\pi=\frac{3\sqrt{3}-\pi-4\pi+2\pi\sqrt{3}}{12}=\frac{3\sqrt{3}-5\pi+2\pi\sqrt{3}}{12}$ A1A1

**c** The length $CB=1-\left(\sqrt{3}-1\right)=2-\sqrt{3}$ A1

the length $AC=\left(A\hat{O}_1C\right)\times\left(AO_1\right)=\frac{\pi}{6}$ M1A1

Similarly, $AB=\frac{\pi}{6}\times(\sqrt{3}-1)$ A1

Therefore the perimeter is $2-\sqrt{3}+\frac{\pi}{6}+\frac{\pi\sqrt{3}}{6}-\frac{\pi}{6}=2-\sqrt{3}+\frac{\pi\sqrt{3}}{6}$ M1A1

# 9 Modelling with matrices: storing and analyzing data

**1** Markov chains can model transitions in employment patterns over generations of families. Taking one classification of employment as professional (P), white collar (W) and blue collar (B), and assuming each family has one child, the employment classification of the child may transition from the employment classification of his/her mother as follows:

Employment classification of mother

Employment classification of child

$$T = \begin{array}{c|c} & \begin{array}{ccc} \text{P} & \text{W} & \text{B} \end{array} \\ \begin{array}{c} \text{P} \\ \text{W} \\ \text{B} \end{array} & \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.4 \\ 0.1 & 0.2 & 0.5 \end{pmatrix} \end{array}$$

a Find $T^2$ and interpret the entry in the third column, first row.

b Given the initial state matrix $\begin{pmatrix} 231 \\ 651 \\ 2089 \end{pmatrix}$ Predict the employment classifications after 3 generations.

**Exam-style questions**

**2** The matrix $A_n$ is given by $A_n = \begin{pmatrix} 12n & 1-4n \\ 4n+1 & 2n \end{pmatrix}$ for $n \geq 1,\ n \in \mathbb{Z}^+$

Show that $A_n$ is non-singular (4)

**3** **a** Write the following system of equations in the form $AX = B$ (2)

$3a - 4b + 5c - 10d = 0$

$2a + 10b - c + 3d = 21$

$17a + 8b - 13d = 9$

$14b - 3c + 3d = -1$

**b** Hence or otherwise find a solution to the system of equations. (5)

**4** A linear transformation is defined by the matrix $A = \begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix}$

**a** Find the equation of the new line when the line $y = mx$ is transformed by $A$. (4)

**b** Hence find the value of $m$ such that the new line is vertical. (2)

**5** A linear transformation is defined by the matrix $T = \begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix}$

**a** Find the eigenvalues and eigenvectors of the matrix $T$. (7)

**b** Hence write down the equations of two lines that are invariant under the transformation $T$. (2)

**6** A small town contains two supermarkets: Texo and Whiterose.

Initially, it was determined that Texo had 1500 regular customers, while Whiterose had 1200.

Analysis of customers over a particular month found that $30\%$ of Texo customers switched to shopping Whiterose, while only $16\%$ of Whiterose shoppers switched to Texo.

**a** By forming a transition matrix, find the expected market share for Texo and Whiterose after two months have passed. (6)

**b** Find the long-term prediction of market share for each supermarket. (4)

**7** An IB school grades their HL maths students A, B, C, D according to their level of ability. Each semester it tests their students to see if there have been any improvements (or otherwise) in their grades.

The transition matrix $T = \begin{pmatrix} 0.85 & 0.15 & 0.03 & 0.02 \\ 0.10 & 0.70 & 0.08 & 0.05 \\ 0.03 & 0.10 & 0.82 & 0.17 \\ 0.02 & 0.05 & 0.07 & 0.76 \end{pmatrix}$ represents the probability of an individual student transitioning from one grade to another over the course of a single semester.

One year consists of three semesters. A student is said to 'improve' if he or she maintains at least the same level of grade that he or she had at the start of the year.

**a** Determine the percentage of grade A students who have improved. (4)

**b** Determine the percentage of grade B students who have improved. (2)

**c** Determine the percentage of grade C students who have improved. (2)

**d** Determine the percentage of grade D students who have improved. (2)

**e** Find the steady state matrix $T$. (3)

**f** The school claims to be an 'improving' school with respect to the results of their HL mathematics students.
Comment on the extent to which the school's claim be justified. (3)

**8** Shadowshore and Goldbeach are two coastal towns.

Each year, 2% of Shadowshore's population move to Goldbeach, and 5% of Goldbeach's population moves to Shadowshore.

The initial population of Shadowshore is 38 000 and the initial population of Goldbeach is 55000.

**a** Write down a transition matrix $T$ showing the population changes between the two towns in one year. (2)

**b** Find the population of each town after three years. (3)

**c** Find a matrix $P$ (with integer elements) that diagonalises $T$. (7)

**d** Hence show that the long term transition matrix $T^{\infty}$ (7)

is given by $T^{\infty} = \begin{pmatrix} \frac{5}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{2}{7} \end{pmatrix}$

**e** Hence determine the expected populations of Shadowshore and Goldbeach over the long term. (3)

**Answers**

**1** **a** $\begin{pmatrix} 0.54 & 0.28 & 0.2 \\ 0.3 & 0.48 & 0.46 \\ 0.16 & 0.24 & 0.34 \end{pmatrix}$ the probability that the grandchild of a bluecollar worker will be a professional is 0.2

**b** $\begin{pmatrix} 866.264 \\ 1311.98 \\ 792.756 \end{pmatrix}$ 866.264 professional, 1311.98 white collar and 792.756 blue collar.

**2** $\det A_n = (12n \times 2n) - (1-4n)(4n+1) = 24n^2 - (1-16n^2)$ M1A1

$= 40n^2 - 1$ A1

Since $\det A_n \neq 0$ for all $n \geq 1$, $A_n$ is non-singular R1

**3**

**a** $\begin{pmatrix} 3 & -4 & 5 & -10 \\ 2 & 10 & -1 & 3 \\ 17 & 8 & 0 & -13 \\ 0 & 14 & -3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 21 \\ 9 \\ -1 \end{pmatrix}$ M1A1

**b** Using GDC: M1

$a = \dfrac{3929}{958}$ A1

$b = \dfrac{553}{958}$ A1

$c = \dfrac{7715}{958}$ A1

$d = \dfrac{4815}{958}$ A1

**4** **a** $\begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} (4-m)x \\ 3mx \end{pmatrix}$ M1A1

The new line is therefore $Y = \dfrac{3m}{4-m}X$ M1A1

**b** The new line is vertical if the gradient is undefined; i.e. when $m = 4$ M1A1

**5** **a** To solve $Ax = \lambda x$ :

$\begin{vmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{vmatrix} = 0$ M1

$(1-\lambda)^2 - 36 = 0$ A1

$\lambda = -5$ or $\lambda = 7$ A1A1

When $\lambda = -5$, $x = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$ (or an integer multiple thereof) M1A1

When $\lambda = 7$, $x = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$ (or an integer multiple thereof) A1

**b** $y = \frac{2x}{3}$ and $y = -\frac{2x}{3}$ A1A1

**6** **a** $T = \begin{pmatrix} 0.7 & 0.16 \\ 0.3 & 0.84 \end{pmatrix}$ M1

$\begin{pmatrix} 0.7 & 0.16 \\ 0.3 & 0.84 \end{pmatrix}^2 \begin{pmatrix} 1500 \\ 1200 \end{pmatrix}$ M1A1

$= \begin{pmatrix} 1103 \\ 1597 \end{pmatrix}$ A1

Texo: $\frac{1103}{1700} \times 100 = 64.9\%$ A1

Whiterose: $100 - 64.9 = 35.1\%$ A1

**b** The steady state matrix is $\begin{pmatrix} 0.35 & 0.35 \\ 0.65 & 0.65 \end{pmatrix}$ M1A1

The long-term prediction would therefore be for

Texo to have 35% of customers, A1

and Whiterose to have 65%. A1

**7** **a** $T^3 = \begin{pmatrix} 0.65 & 0.28 & 0.10 & 0.07 \\ 0.19 & 0.40 & 0.16 & 0.19 \\ 0.10 & 0.21 & 0.60 & 0.33 \\ 0.06 & 0.11 & 0.14 & 0.47 \end{pmatrix}$ M1A1A1

So 65% A1

**b** 68% M1A1

**c** 86% M1A1

**d** 100% M1A

**e** $T^\infty = \begin{pmatrix} 0.30 & 0.30 & 0.30 & 0.30 \\ 0.21 & 0.21 & 0.21 & 0.21 \\ 0.32 & 0.32 & 0.32 & 0.32 \\ 0.16 & 0.16 & 0.16 & 0.16 \end{pmatrix}$ M1A1A1

**f** Over the course of one year, the majority of students at least maintain their initial grade, so the school could reasonably be said to improve. R1

However, long term, it is only the B, C, D graded students that improve, generally at the expense of the grade A students. R1

So taking a long-term view, the school would only partially be justified in their claim. R1

**8** **a** $T = \begin{pmatrix} 0.98 & 0.05 \\ 0.02 & 0.95 \end{pmatrix}$ M1A1

**b** $T^3 = \begin{pmatrix} 0.98 & 0.05 \\ 0.02 & 0.95 \end{pmatrix}^3 = \begin{pmatrix} 0.944 & 0.140 \\ 0.056 & 0.860 \end{pmatrix}$ M1A1

$\begin{pmatrix} 0.944 & 0.140 \\ 0.056 & 0.860 \end{pmatrix}\begin{pmatrix} 38000 \\ 55000 \end{pmatrix} = \begin{pmatrix} 43562 \\ 49438 \end{pmatrix}$ A1

**c** To solve $Ax = \lambda x$ :

$\begin{vmatrix} 0.98-\lambda & 0.05 \\ 0.02 & 0.95-\lambda \end{vmatrix} = 0$ M1

$(0.98-\lambda)(0.95-\lambda) - 0.05 \times 0.02 = 0$

$\lambda = 0.93$ or $\lambda = 1$ A1A1

When $\lambda = 0.93$, $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or an integer multiple thereof) M1A1

When $\lambda = 1$, $x = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ (or an integer multiple thereof) A1

Therefore $P = \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}$ A1

**d** $P^{-1} = \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}$ A1

$D = \begin{pmatrix} 0.93 & 0 \\ 0 & 1 \end{pmatrix}$ A1

$T^n = PD^nP^{-1} = \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0.93^n & 0 \\ 0 & 1^n \end{pmatrix}\frac{1}{7}\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}$ M1A1

$= \frac{1}{7}\begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0.93^n & 0 \\ 0 & 1^n \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}$

As $n \to \infty$ , $D^n = \begin{pmatrix} 0.93^n & 0 \\ 0 & 1^n \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ M1A1

So $T^n \to \frac{1}{7}\begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{2}{7} \end{pmatrix}$ A1

**e** $\begin{pmatrix} \frac{5}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{2}{7} \end{pmatrix}\begin{pmatrix} 38000 \\ 55000 \end{pmatrix} = \begin{pmatrix} 66428 \\ 26571 \end{pmatrix}$ M1

So the populations over the long-term will be

Shadowshore 66428 A1

Goldbeach 26571 A1

# 10 Analyzing rates of change: differential calculus

**1** In Physics, "jerk" is the rate of change of acceleration and "jounce" is the rate of change of jerk. Complete the table:

| | | | | | |
|---|---|---|---|---|---|
| $s(t)$ | $t^5$ | $\sqrt{t}$ | $\ln(2t)$ | $-\cos(3t)$ | |
| $v(t)$ | | | | | |
| $a(t)$ | | | | | $e^{-0.2t}$ |
| jerk$(t)$ | | | | | |
| jounce$(t)$ | | | | | |

**2** **a** Prove that $y = \dfrac{x^2+5}{x-2}$ has two stationary points and state their nature.

**b** Prove that $y = \dfrac{x^2-5}{x-2}$ has no stationary points.

**3** Let P be the point (1,1,1) and let Q lie on the line with parametric equation $x = 2-\lambda$ , $y = 3$ , $z = 3-3\lambda$

**a** Find the vector $\overrightarrow{\mathrm{PQ}}$ in terms of $\lambda$

**b** Hence the shortest distance from the point to the line. Give your answer correct to five significant figures.

**4** A satellite's orbits the earth according to the equation $r = 6.1+1.8\mathrm{e}^{-0.5t}$ where $r$ is its distance from the centre of the earth measured in units equal to the radius of the earth and $t$ is measured in days.

**a** State the distance from the earth when $t = 0$

The satellite is slowly descending into a stable orbit.

**b** Find the satellite's distance from the earth when it reaches the stable orbit.

**c** Find the rate of change of the satellite's distance from the earth at time $t$.

**d** Find the value of $t$ at which the magnitude of the rate of change first falls below 0.1 units/day.

**5** Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings. This can be written as $\dfrac{\mathrm{d}T}{\mathrm{d}t} = k\left(T - T_s\right)$ where $T$ is the temperature of the object, $t$ is the time elapsed and $T_s$ is the temperature of the surroundings.

**a** Use differentiation to verify that the equation $T = Ae^{kt} + T_s$ , where $A$ is a constant, satisfies Newton's law of cooling.

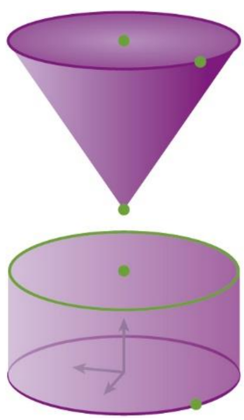
**b** A bottle containing water with temperature $10^\circ C$ is taken from an air conditioned hotel room onto the beach where the temperature is $35^\circ C$ at 11.00 am. After 5 minutes the water has temperature $13^\circ C$. At what time will the temperature of the bottle of water reach $25^\circ C$? Give your answer to the nearest minute.

**c** Newton's law of cooling can be applied in police investigations as follows:

Police are called to an office building where a body has been found. They inspect the scene at 0917 and determine that the temperature of the body at that time is $30^\circ C$. Inquires reveal that the office temperature is regulated at $17^\circ C$. The temperature of the body at 1217 is found to be $26^\circ C$. Estimate the time of death to the nearest minute if the normal living body temperature is $37^\circ C$

**6** A perfume company wishes to design a bottle in the shape of a square based right pyramid. The bottle must hold 100 $cm^3$ . What are the dimensions of the pyramid with the least surface area?

**7** A coffee percolator in Chancer's café takes the form of an inverted cone coffee filter and a cylindrical jug as shown in this diagram:



The coffee drains from the cone at a rate of 10 cm/s and collects in the jug, both of which have base diameter 10 cm. The height of the cone is 15 cm.

**a** What must the height of the cylinder be in order for there to be no waste of coffee?

**b** At the instant when the depth of the coffee in the cone is 5 cm, find the rate at which the depth of coffee in the cone is changing.

**c** At this instant, what is the rate of change of the depth of coffee in the jug?

**d** Matt the barista wonders if there is ever a time when the rate of change of both the depth in the cone and in the jug are identical in magnitude. Solve this problem for him.

**8** **a** Find the value of $x$ which minimises the function $T_1(x) = \dfrac{\sqrt{25+(12-x)^2}}{70} + \dfrac{x}{110}$.

**b** Find the value of $\theta$ which minimises the function $T_2(\theta) = \dfrac{5}{70\sin(\theta)} + \dfrac{12 - \dfrac{5}{\tan(\theta)}}{110}$.

**c** Show that these values solve the problem of finding the shortest journey from Alphaville to Betatown in Chapter one review exercise question 11 from two different yet consistent perspectives.

**Exam-style questions**

**9 a** Write down $\frac{d}{dx}\sin x$ and subsequently find the derivatives of

**i** $x\sin x$ **ii** $\sin\left(\frac{1}{x}\right),\ x \neq 0$ **iii** $x\sin\left(\frac{1}{x}\right),\ x \neq 0$. (6)

**b** Differentiate $e^x\cos x$ and find the stationary points of $y = e^x\cos x$. Sketch the curve $y = e^x\cos x$. (6)

**c i** By sketching the functions $y = -x$ and $y = \tan x$ on the same axes, show that $y = x\sin x$ has one stationary point in each interval

$$\frac{\pi}{2} + n\pi < x < \frac{\pi}{2} + (n+1)\pi,\ n = \ldots, -2, -1, 0, 1, 2, \ldots.$$

**ii** Show that $f(x) = x\sin x$ is even ( $f(x) = f(-x)$ ) and sketch $y = x\sin x$. (5)

**10** Jerry and Amy baked 100 chocolate muffins to sell at school as part of their plan to raise money for the graduation trip. The profit ($P$) in Euros made by the two students selling the homemade muffins is modelled by the function $P(x) = -\frac{1}{40}x^2 + 3x - 10$, where $x$ is the number of muffins sold.

**a** State

**i** the value of $P(0)$

**ii** the meaning of $P(0)$ in the context of the problem

**iii** the largest possible domain of the function $P$. (4)

**b** Draw axes for $x$ and $P$, placing $x$ on the horizontal axis and $P$ on the vertical axis. Use suitable scales. Sketch the graph of $P$ against $x$. Label your graph. (4)

**c** Use your graph to find

**i** the maximum possible profit

**ii** the number of muffins that need to be sold to make the maximum profit

**iii** the minimum number of muffins that need to be sold to make a profit of 60 euros. (3)

The two students sold 80 muffins. They decide to deposit the money in a bank account for 6 months. The local bank offers them 4% interest rate, compounded monthly.

**d** Calculate the interest they earn after 6 months. Give your answer to the nearest cent. (4)

**11** A cup of tea is left on a table for several hours. Initially its temperature was 94 °C. After 20 minutes, the temperature had dropped to 29°.

The temperature of the tea in the cup is modelled by $T(t) = 25 + ae^{bt}$, where $t$ represents the time in minutes that the cup of tea has been on the table.

**a** Find the value of

i $a$ (2)

ii $b$ (2)

**b** Use the model to estimate the temperature of the tea after it has been sitting on the table for 30 minutes. (2)

**c** Write down the equation of the horizontal asymptote of the graph of $T$. (1)

**d** State the meaning of the asymptote found in (b). (1)

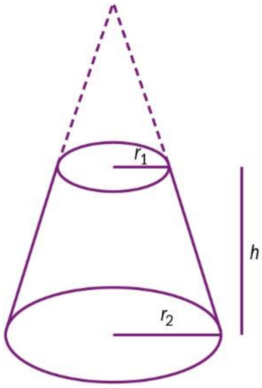
**12 a** The logistic function has the standard form $f(x) = \frac{L}{1+Ce^{-kx}}$. Find the derivative $f'(x)$.

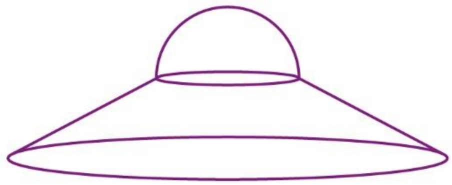
(3)

**b** The logistic function is often used to model population growth. A population is modelled by $N(t) = \frac{100}{1+49e^{-0.01x}}$. Estimate the size of the population and its rate of change when $t = 0$ and $t = 100$. Write down the limiting population if the model is valid for all $t$. (6)

**13 a** Show that the volume of a truncated cone of height $h$ and radii $r_1$ and $r_2$ (see the diagram below) is $V = \frac{1}{3}\pi h\left(r_1^2 + r_1 r_2 + r_2^2\right)$. [Hint: consider the volume of the 'missing' cone and relate the height of this cone to $h$, $r_1$ and $r_2$] (7)

**b** A flying saucer spaceship is made up of a hemisphere (radius $r$) on top of a truncated cone (height $h$, radii $r$ and $4r$). Find the volume of the spaceship. Suppose the height of the entire spaceship is fixed at a constant value, $C$. Find the maximum volume of the spaceship in terms of $C$, and show that it is a maximum.



(10)

**Answers**

**1**

| | | | | | |
|---|---|---|---|---|---|
| $s(t)$ | $t^5$ | $\sqrt{t}$ | $\ln(2t)$ | $-\cos(3t)$ | $25e^{-0.2t}$ |
| $v(t)$ | $5t^4$ | $\frac{1}{2}t^{-\frac{1}{2}}$ | $\frac{1}{t}$ | $3\sin(3t)$ | $-5e^{-0.2t}$ |
| $a(t)$ | $20t^3$ | $\frac{-1}{4}t^{-\frac{3}{2}}$ | $\frac{-1}{t^2}$ | $9\cos(3t)$ | $e^{-0.2t}$ |
| jerk$(t)$ | $60t^2$ | $\frac{3}{8}t^{-\frac{5}{2}}$ | $\frac{2}{t^3}$ | $-27\sin(3t)$ | $-0.2e^{-0.2t}$ |
| jounce$(t)$ | $120t$ | $\frac{-15}{16}t^{-\frac{5}{2}}$ | $\frac{-6}{t^4}$ | $-81\cos(3t)$ | $0.04e^{-0.2t}$ |

**2** **a** (5,10) is a local minimum (-1,-2) is a local maximum.

**b** Proof

**3** **a** $\begin{pmatrix} 1-\lambda \\ 2 \\ 2-3\lambda \end{pmatrix}$

**b** 2.0248

**4** **a** 7.9 units

**b** 6.1 units

**c** $\dot{r} = -0.9e^{-0.5t}$

**d** 4.39 days

**5** **a** Proof **b** 63 minutes **c** 0546

**6** Height = 8.43, side of square base = 5.96.

**7** **a** 5cm **b** $\frac{-18}{5\pi}$ *cm / sec* **c** $\frac{2}{5\pi}$ *cm / sec*

**d** It can't happen unless the base radius of the cone was 15cm.

**8** **a** 7.88 **b** 50.5 degrees **c** Proof

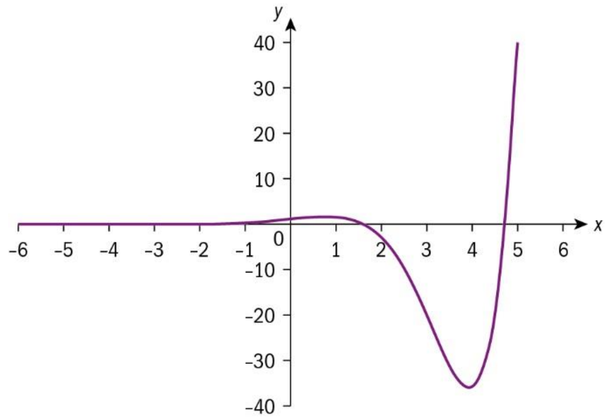
**9** **a** $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$ A1

**i** $\frac{\mathrm{d}}{\mathrm{d}x}x\sin x = x\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) + \sin x\frac{\mathrm{d}}{\mathrm{d}x}(x) = x\cos x + \sin x$ M1A1

**ii** $u = \frac{1}{x},\ \frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{1}{x}\right) = \frac{\mathrm{d}}{\mathrm{d}u}(\sin u)\times\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$ M1A1

**iii** $\frac{\mathrm{d}}{\mathrm{d}x}x\sin\left(\frac{1}{x}\right) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\frac{1}{x}\right)\right) + \sin\left(\frac{1}{x}\right)\frac{\mathrm{d}}{\mathrm{d}x}(x) = -\frac{1}{x}\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$ A1

**b** $\frac{\mathrm{d}}{\mathrm{d}x}e^x\cos x = e^x\frac{\mathrm{d}}{\mathrm{d}x}\cos x + \cos x\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x\cos x - e^x\sin x$ M1A1




$y = e^x\cos x,\ y' = e^x\cos x - e^x\sin x = 0$

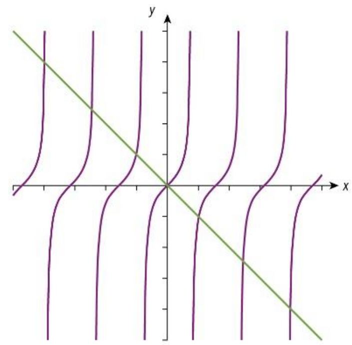
$\Rightarrow \cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4} + n\pi$ M1A1

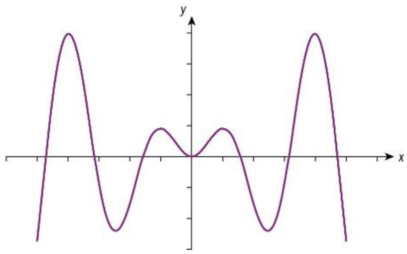
Figure should be labelled with the value of $y\ (=1)$ when $x = 0$ and show that this isn't a maxima but rather the closest maxima is at $x = \frac{\pi}{4}$ M1A1

**c** **i** $y = x\sin x,\ \frac{\mathrm{d}y}{\mathrm{d}x} = x\cos x + \sin x = 0 \Rightarrow -x = \tan x$ M1

The first sketch below shows $y = -x$ (orange) and $y = \tan x$ (blue). The vertical asymptotes of $y = \tan x$ are given by $x = \frac{\pi}{2} + n\pi$. The two functions intersect once in every interval $\frac{\pi}{2} + n\pi < x < \frac{\pi}{2} + (n+1)\pi,\ n = \ldots, -2, -1, 0, 1, 2, \ldots$ and the intersection gets closer to the vertical asymptote as $x$ increases. M2A1

**ii** $f(x)=x\sin x,\ \ f(-x)=(-x)\sin(-x)=(-x)(-\sin x)=x\sin x$. The second sketch



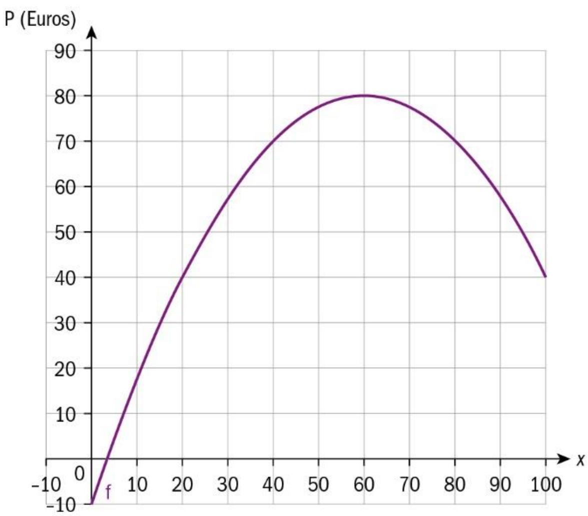

shows $y=x\sin x$. M1A1

**10 a i** $P(0)=-10$ A1

**ii** the cost of making the muffins. A1

**iii** $0\le x\le 100$ A2

**b**




(shape: A1; *y*-intercept: A1; axes labels: A1 mark; domain: A1)

**c i** 80 euros A1

**ii** 60 muffins A1

**iii** 32 muffins A1

**d** They earn 70 Euros profit A1

Interest $=70\left(1.04^{6}-1\right)$ M1A1

$=18.57$ Euros A1

**11 a** **i** $T(0)=94\Rightarrow 25+a=94$ M1

$a=69$ A1

**ii** $T(20)=29\Rightarrow 25+69\mathrm{e}^{20b}=29$ M1

$b=-0.142$ (3 s.f.) A1

**b** $T(30)=25+69\mathrm{e}^{-0.142\times 30}=26.0\,°\mathrm{C}$ (3 s.f.) M1A1

**c** $T=25$ A1

**d** The temperature of the room. R1

**12 a** $f(x)=\dfrac{L}{1+Ce^{-kx}}=L\left(1+Ce^{-kx}\right)^{-1}=Lu^{-1}$ (or use of quotient rule) M1

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)=f'(x)=\frac{\mathrm{d}}{\mathrm{d}u}\frac{L}{u}\times\frac{\mathrm{d}}{\mathrm{d}x}\left(1+Ce^{-kx}\right)=\frac{-L}{\left(1+Ce^{-kx}\right)^{2}}\left(-Cke^{-kx}\right)=\frac{LCke^{-kx}}{\left(1+Ce^{-kx}\right)^{2}}$$ (M1,A1)

**b** $N(t)=\dfrac{100}{1+49e^{-0.01x}}$

Using part **a** with $L=100, C=49, k=0.01$ gives $N'(t)=\dfrac{49e^{-0.01x}}{\left(1+49e^{-0.01x}\right)^{2}}$ M1A1

(M1)

$N(0)=2,\ N'(0)=\dfrac{49}{50^{2}}=0.0196$ (A1)

$$N(100)=\frac{100}{1+49e^{-1}}=5.255\text{ (3 d.p.)},\ N'(100)=\frac{49e^{-1}}{\left(1+49e^{-1}\right)^{2}}=0.050\text{ (3 d.p.)}$$

Assuming the population is discrete (integer valued), accept 5 for N(100). (A2)

As $t\to\infty,\ N(t)\to 100$. (R1)

**13 a** Let the height of the 'missing' cone (given by the dotted lines) be *H*. Similar triangles give

$$\frac{r_2}{H+h}=\frac{r_1}{H}\Rightarrow r_2H=r_1h+r_1H\Rightarrow H=\frac{r_1h}{r_2-r_1},\ H+h=\frac{Hr_2}{r_1}$$ (M2A1)

The volume of the truncated cone is the difference between the larger cone (made up of the truncated cone and the 'missing' cone together), of radius $r_2$ and height *H*+*h*, and the 'missing' cone of radius $r_1$ and height *h*. R1

$$V=\frac{1}{3}\pi(h+H)r_2^2-\frac{1}{3}\pi hr_1^2=\frac{1}{3}\pi\left(\frac{Hr_2}{r_1}\right)r_2^2-\frac{1}{3}\pi Hr_1^2=\frac{1}{3}\pi\left(\frac{r_2^3-r_1^3}{r_1}\right)H$$

$$=\frac{1}{3}\pi\left(\frac{r_2^3-r_1^3}{r_1}\right)\frac{r_1h}{r_2-r_1}=\frac{1}{3}\pi h\left(\frac{r_2^3-r_1^3}{r_2-r_1}\right)=\frac{1}{3}\pi h\left(r_1^2+r_1r_2+r_2^2\right)$$ (M2,A1)

**b** $V=\frac{1}{3}\pi h\left(r^2+r\times 4r+16r^2\right)+\frac{2}{3}\pi r^3=7\pi hr^2+\frac{2}{3}\pi r^3$ M1A1

$h=C-r,\ \ V=7\pi(C-r)r^2+\frac{2}{3}\pi r^3=7\pi Cr^2-\frac{19}{3}\pi r^3$ M1A1

$\frac{\mathrm{d}V}{\mathrm{d}r}=14\pi Cr-19\pi r^2=0\Rightarrow r=0,\ r=\frac{14C}{19}$. We are only interested in $r>0$ so $r=\frac{14C}{19}$

(M1,A1)

$V=7\pi C\left(\frac{14C}{19}\right)^2-\frac{19}{3}\pi\left(\frac{14C}{19}\right)^3=\frac{1372\pi C^3}{1083}$ (A1)

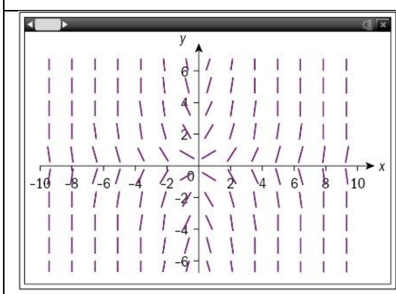
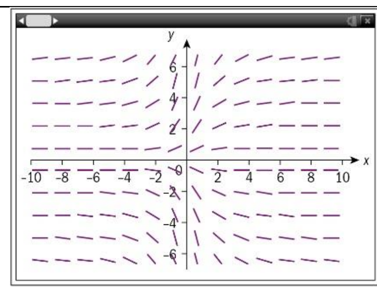
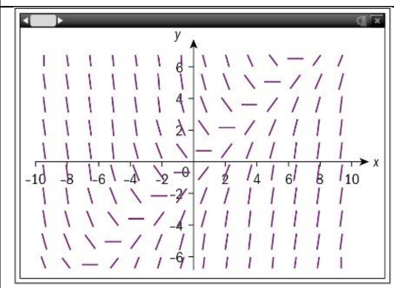
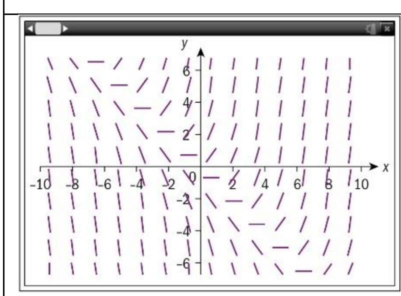
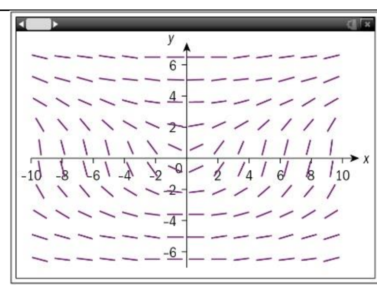
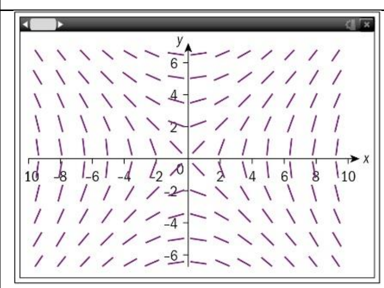
$\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=14\pi C-38\pi r,\ r=\frac{14C}{19}\Rightarrow\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=-14\pi C<0$ so a maximum. M1A1R1

# 11 Approximating irregular spaces: integration and differential equations

**1** Match the differential equation with the correct slope field:

**a** $\dfrac{dy}{dx} = x - y$ **b** $y' = x + y$ **c** $\dfrac{dy}{dx} = \dfrac{x}{1+y^2}$ **d** $y' = xy$

**e** $\dfrac{dy}{dx} = \dfrac{y}{1+x^2}$

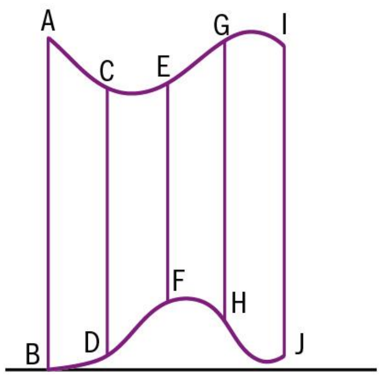
| P | Q | R |
|---|---|---|
|  |  |  |
| S | T | U |
|  |  |  |

**2** Identify which of the following integrals can be found by inspection (I), substitution (S) or require technology (T) and then find their values, stating any substitution used:

**a** $\int_0^1 x\sqrt{1+x}\ dx$ **b** $\int_0^1 x^2\sqrt{1+x}\ dx$ **c** $\int_0^1 \sqrt{x}\left(1+x^2\right)dx$

**d** $\int_0^1 x\sqrt{1+x^2}\ dx$ **e** $\int_0^1 x(1+x^2)\ dx$ **f** $\int_0^1 x^2\sqrt{1+x^3}\ dx$

**3** Iana is designing a padding pool for her garden bounded by the lines AB and IJ and two curves from A to I and B to J as shown. Iana makes measurements of the vertical lines AB=11m, CD=8m, EF=6m, GH=8m and IJ=10m. Each vertical line is 2.5 m apart.




Iana also draws a horizontal axis as shown and measures the distances from the points B, D, F, H and J to this axis as 0, 0.5, 3, 2 and 0.5 metres respectively.

Apply the trapezoidal rule to estimate the volume of water needed to fill Iana's pool given that the depth of the water will be a constant value of 0.5 m.

**4** Marin is designing parts for an aircraft engine and experiments with the choice of axis of revolution.

He first rotates the region formed by the graph of $g(x) = 2 + \frac{1}{x-2}$, the $x$ – axis and the lines $x$ = 3 and $x$ = 4 through $2\pi$ radians about the $x$ – axis.

**a** Find the volume of the solid of revolution generated.

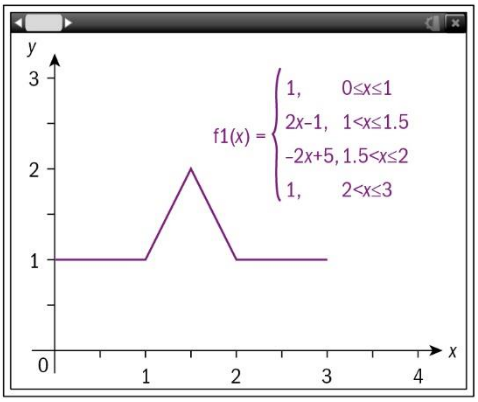
**b** Marin then rotates the curve of $x = g^{-1}(y)$ for $3 \le y \le 4$ through $2\pi$ radians about the $y$ – axis. Find the volume of the solid of revolution generated.

**c** Write down and explain the relationship between your answers for **a** and **b**.

**d** Marin then states that if he rotates the function

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 2x-1 & 1 < x \le 1.5 \\ -2x+5 & 1.5 < x \le 2 \\ 1 & 2 < x \le 3 \end{cases}$$

around the $x$ –axis it will give the same volume as rotating it around the $y$ – axis. Determine if Marin's statement is correct by calculating both volumes of revolution.




**5** A robot welding unit moves along a straight line in a car factory with acceleration in cm s$^2$ at time $t$ seconds, $t \ge 0$ given by the formula $a = -\frac{1}{3}v + 1$. When $t$ = 0, the velocity is 34 cm s$^2$.

**a** Find an expression for $v$ in terms of $t$.

**b** Hence calculate the distance travelled by the robot between $t$ = 1 and $t$ = 2 seconds.

**6** A teardrop shaped earring is being re-designed. The graph of $y=\frac{x}{a}\sqrt{c-x^2}$ , $0\le x\le b$ where $x$ and $y$ are measured in mm is rotated by $2\pi$ radians around the $x$ – axis. The manufacturers require that the volume of the earring be 1000 $mm^3$ . Complete the table to give three possible solutions to this design problem.

| a | b | c |
|---|---|---|
| | 40 | 1600 |
| 100 | | 1000 |
| 100 | <30 | |

**7** The shape of a flower garden is planned as the shape enclosed by the graphs of $f(x)=e^x$ and $g(x)=\frac{1}{\cos^2(x)}$ where $0\le x\le 1$.

**a** Use technology to sketch the shape of the flower garden.

**b** Find the coordinates of the points of intersection of the functions $f(x)=e^x$ and $g(x)=\frac{1}{\cos^2(x)}$ where $0\le x\le 1$

**c** Hence calculate the area of the flower garden.

**d** The designer wishes to build a walkway above the flower garden, bounded by the graphs of $y=f(x)$ and $y=g(x)$ and the lines $x = c$ and $x = c + 0.2$ where $0\le c\le 0.664$. Find the value of $c$ which maximizes the area of the walkway.

**8** Two drones A and B start journeys at the same time to survey areas under threat from a forest fire. The initial displacements and initial velocities of both drones are -3m and 0 m s$^{-1}$ respectively. The acceleration of drone A is $a_A=-4\sin 2t$ and of drone B $a_B=3\cos 3t$ .

**a** Find the position of each drone after 5 seconds.

**b** Find in the first 7 seconds of flight the times when the drones are further apart and the distance between them at these times.

**9** Teodora models the design of the cross sectional area of a walkway in a new airport terminal with the general form of a function $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ , where $\sigma,\mu\in\mathbb{R}^+$

**a** Show that $f'(x)=-f(x)\times\left(\frac{x-\mu}{\sigma^2}\right)$ is true in general.

**b** Hence show that the turning point of $f(x)$ has coordinates $\left(\mu,\frac{1}{\sigma\sqrt{2\pi}}\right)$ and that the points of inflexion have $x$ - coordinates $\left(\mu+\sigma,\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}}\right)$ and $\left(\mu-\sigma,\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}}\right)$

**c** Use technology to sketch the graph of one of the models Teodora considers a vertical stretch of the general form: $f(x)=\frac{100}{3\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-10}{3}\right)^2}$

Hence predict:

**d** the coordinates of the points on the roof where the gradient is steepest

**e** the value of $d$ such that the lines $x$ = 10+$d$ and $x$ = 10 - $d$ and the graph of $f(x)=\frac{100}{3\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-10}{3}\right)^2}$ enclose an area of 68% of the total area of the cross section.

**Exam-style questions**

**10 a** Apply the trapezium rule with a strip width of 1 to find the integral $\int_{-2}^{2} x^3+8\,dx$ . (3)

**b** Find the indefinite integral $\int x^3+8\,dx$ and hence find the exact value of $\int_{-2}^{2} x^3+8\,dx$ . (3)

**c** Comment on your answers to parts **a** and **b**, and explain why this is the case. Use a sketch to assist your explanation. (3)

**11** Given that $\int_1^k \frac{1}{x}dx=\int_4^{12}\frac{1}{2x+1}dx$ , find the exact value of the positive constant $k$. (7)

**12** An object starts from rest and moves in a straight line. Its acceleration is given by $a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=\cos t$, where distance $x$ is in metres and time $t$ is in seconds. Angles are given in radians.

**a** Find the object's velocity $v$ as a function of $t$.
(3)

**b** Hence, find the total distance travelled by the object in the first 10 seconds. (4)

**13** A friendly bacteria is multiplying. Its rate of growth is given by the differential equation $\frac{dx}{dt}=\frac{xt}{5}$ , where $x$ is the number of bacteria and $t$ is time in days. Initially $x=3$.

**a** Solve the differential equation, giving $x$ as a function of $t$ . (7)

**b** Calculate the number of bacteria when $t=7$. (2)

**c** Find how long it takes before the number of bacteria exceeds one million. (3)

**14 a** The portion of the curve $y=x^2$ from $x=0$ to $x=2$ is rotated through $2\pi$ about the $x$ axis. Find the exact value of the volume of the solid of revolution that is created (it looks like the end of a trumpet). (4)

**b** The portion of the curve $y=x^2$ from $x=0$ to $x=2$ is rotated through $2\pi$ about the $y$ axis. Find the exact value of the volume of the solid of revolution that is created (it looks like a bowl). (4)

c Now consider the portion of the curve $y=x^2$ from $x=0$ to $x=t,(t>0)$ being rotated through $2\pi$ about (i) the $x$ axis (ii) the $y$ axis. Find the value of $t$ so that the two volumes of the solids of revolution are equal. Hence, state the range of values of $t$ for which the volume about the $x$ axis will be greater than the volume about the $y$ axis. (7)

**Answers**

**1** **a** R **b** S **c** T **d** P

**e** Q

**2** **a** T or S, $u = 1 + x$, 0.644

**b** T, 0.440 **c** I, $\frac{20}{21}$ **d** S, $u = 1 + x^2$, 0.609

**e** I, $\frac{3}{4}$ **f** S, $u = 1 + x^3$, 0.406

**3** 4.09 $m^3$

**4** **a** 22.8 units cubed

**b** 22.8 units cubed

**c** the volumes are equal because the function and the limits of integration have been reflected in the line $y = x$.

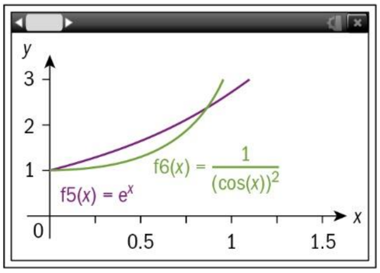
**d** Around the $x$ – axis: 13.6 units cubed, Around the $y$ – axis: 28.3 units cubed.

**5** **a** $v = 31e^{\frac{-t}{3}} + 3$ **b** 21.9 cm

**6**

| | | |
|---|---|---|
| a | b | c |
| 207 | 30 | 900 |
| 100 | 24.7 | 900 |
| 100 | $\sqrt{894} < 30$ | 894 |

**7** **a**




**b** (0,0) and (0.864,2.37)

**c** 0.202

**d** c= 0.453

**8** **a** Drone A is 13.54 m to the left of the start and drone B is 2.41 m to the left of the start.

**b** At 5.76 seconds the drones are 12.7 m apart, the maximum distance between them in the first seven seconds

**9** **a** $f'(x)=\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\times-\left(\dfrac{x-\mu}{\sigma}\right)\times\dfrac{1}{\sigma}=-f(x)\times\left(\dfrac{x-\mu}{\sigma^2}\right)$

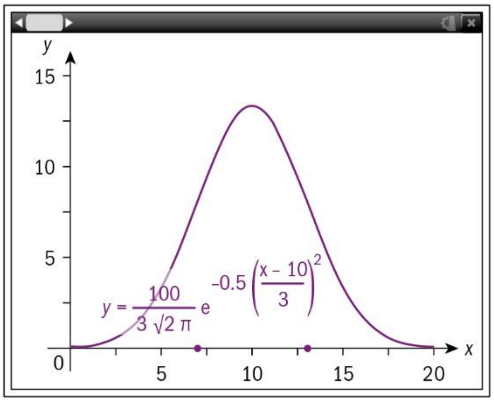
**b** $f'(x)$ changes sign at $x=\mu$ from positive to negative hence $\left(\mu,\dfrac{1}{\sigma\sqrt{2\pi}}\right)$ is a maximum.

$f''(x)=-f'(x)\times\left(\dfrac{x-\mu}{\sigma^2}\right)-f(x)\times\left(\dfrac{1}{\sigma^2}\right)=f(x)\times\left(\dfrac{x-\mu}{\sigma^2}\right)^2-f(x)\times\left(\dfrac{1}{\sigma^2}\right)=$

$\dfrac{f(x)}{\sigma^2}\times\left((x-\mu)^2-\sigma^2\right)$ which changes sign at $x=\mu\pm\sigma$ hence $\left(\mu+\sigma,\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}}\right)$ and

$\left(\mu-\sigma,\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}}\right)$ are points of inflexion.

**c**




**d** The gradient is steepest at the inflexion points: (7,8.07) and (13,8.07)

**e** $d$ = 3.

**10 a**

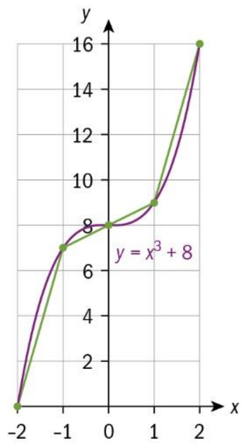
| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 0 | 7 | 8 | 9 | 16 |

Trapezium gives $\dfrac{1}{2}\left(0+2\left(7+8+9\right)+16\right)=32$ A1M1A1

**b** $\dfrac{x^4}{4}+8x+c$ $\left[\dfrac{x^4}{4}+8x\right]_{-2}^{2}=32$ A1M1A1

**c** The trapezium rule calculated the answer completely correctly. A1

For the first two strips the graph is concave down and the trapeziums will miss area, for the next two strips the graph is concave up and the trapeziums will have too much area. By symmetry these errors will cancel in pairs. R1




A1

**11** $[\ln x]_1^k = \left[\frac{1}{2}\ln(2x+1)\right]_4^{12} \Rightarrow \ln k = \frac{1}{2}\ln 25 - \frac{1}{2}\ln 9 = \frac{1}{2}\ln\frac{25}{9} = \ln\frac{5}{3}$ M1A1A1M1A1A1

So $k = \frac{5}{3}$ A1

**12 a** $v = \int \cos t\, dt = \sin t + c$ , $c = 0$ since object start from rest. So $v = \sin t$ M1A1R1

**b** $\int_0^{10} |\sin t|\, dt = 6.16m(3sf)$ M1A1A2

**13 a** $\int \frac{1}{x} dx = \int \frac{t}{5} dt \Rightarrow \ln x = \frac{t^2}{10} + c \Rightarrow x = e^{\frac{t^2}{10}+c} = Ae^{\frac{t^2}{10}}$ M1A1A1M1A1

$t = 0, x = 3 \Rightarrow A = 3$ , $x = 3e^{\frac{t^2}{10}}$ M1A1

**b** $3e^{\frac{49}{10}} \approx 403$ M1A1

**c** Solving $3e^{\frac{t^2}{10}} > 10^6$ gives $t = 11.3$ days. M1A2

**14 a** $\int \pi y^2\, dx = \int_0^2 \pi x^4\, dx = \left[\frac{\pi x^5}{5}\right]_0^2 = \frac{32\pi}{5}$ M1A1A1A1

**b** $\int \pi x^2\, dy = \int_0^4 \pi y\, dy = \left[\frac{\pi y^2}{2}\right]_0^4 = 8\pi$ M1A1A1A1

**c** $\int \pi y^2\, dx = \int_0^t \pi x^4\, dx = \left[\frac{\pi x^5}{5}\right]_0^t = \frac{\pi t^5}{5}$ M1A1

$\int \pi x^2\, dy = \int_0^{t^2} \pi y\, dy = \left[\frac{\pi y^2}{2}\right]_0^{t^2} = \frac{\pi t^4}{2}$ M1A1

$\frac{\pi t^5}{5} = \frac{\pi t^4}{2} \Rightarrow t = \frac{5}{2}$ M1A1

$x$ axis rotation volume will be greater for $t > \frac{5}{2}$ R1

# 12 Modelling motion and change in two and three dimensions

**1** A particle undergoes motion so that its velocity at time $t$ is given by the equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4t\cos\left(t^2\right) \\ 6\sin(3t) + 1 \end{pmatrix}$$

Find an expression for

**a** its acceleration at time $t$

**b** its displacement at time $t$ given that its displacement at $t = 0$ is $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

**2** For the two systems given below

**a** $\dfrac{dx}{dt} = -2y$, $\dfrac{dy}{dt} = 2x$  **b** $\dfrac{dx}{dt} = x + 5y$, $\dfrac{dy}{dt} = x - y$

**i** find the eigenvalues

**ii** find the values of $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ at the point (1,0)

**iii** sketch the trajectory that passes through (1,0)

**iv** comment on the shape of the path traced out by the trajectory.

**3** Three forces are acting on a particle. The forces can be written as

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} a \\ b+1 \\ a \end{pmatrix} \text{ and } \begin{pmatrix} -3b \\ 3a \\ -b \end{pmatrix}$$

Given that the resultant of the three forces is parallel to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Find the values of $a$ and $b$.

**4** A fielder in cricket throws a ball to another fielder who is 10 m away. The ball is thrown from a height of 1.2 m with a velocity of 17 $ms^{-1}$ at an angle of $10^\circ$ to the horizontal.

Find the height of the ball when it reaches the second fielder

**5** A river flows with a velocity of 1.5 $ms^{-1}$. A man can row with a speed of 1.8 $ms^{-1}$. He directs the boat so it is facing $20^\circ$ into the current from the direct line across the river.

**a** Write both these velocities in component form. Take the first component in the direction of the flow of the river and the second as the line perpendicular to the flow of the river.

**b** Hence find the resultant velocity of the boat in the direction

**i** of the flow of the river

**ii** perpendicular to the flow of the river

The river is 20m wide

**c** **i** Use your answer to bii to calculate how long it would take

the boat to cross the river

**ii** Hence find the distance downstream the man will be from his starting point when he reaches the opposite bank.

**6** A population of carp ($x$) in a lake will increase in line with the plant food available. Because carp tend to destroy plant life as they eat, as their population increases the population of perch (y) can decrease. In the absence of carp the population of perch will increase very quickly. For low populations of carp a large population of perch can also affect their growth rate.

The change in population of both fish can be expressed by the following coupled differential equation.

$$\dot{x} = 0.3x - 0.1y$$
$$\dot{y} = -0.2x + 0.4y$$

**a** Find the eigenvalues for the system

**b** Find the general solution to the system of equations

**c** Draw the phase diagram for this system, for both positive and negative values of $x$ and $y$.

**d** Given that if one population reaches 0 it becomes extinct, use your phase diagram to give a condition on the original population sizes, $x_0$ and $y_0$ for the carp, rather than the perch to become extinct

**7** Ben is taking part in a competition in which a coconut is dropped from varying heights and he has to hit it in flight with a wooden ball.

The point at which Ben throws the ball is taken as the origin in a coordinate system. Ben is a distance $d$ from where the coconut would land, and the coconut is dropped from a height $h$ above the point where the ball is thrown.

Ben will throw the ball with velocity $V$ at an angle of $\alpha$ to the horizontal and at the moment the coconut is dropped. Ben would like to know where he should aim in order to hit the coconut.

Let $g$ be the acceleration due to gravity and assume the effects of air resistance can be ignored.

**a** Find the displacement of the coconut from the origin $t$ seconds after it is dropped. Give your answer as a column vector in terms of $d$, $g$ and $h$.

**b** Find the displacement of the ball from the origin $t$ seconds after it is thrown.

**c** Hence show that Ben should always aim for the point where the coconut is released.

**8** A planet's orbit around a star can be considered as a circle. The radius of the circle is taken as 1 astronomical unit.

Let the centre of the orbit be the origin in a coordinate system and the plane of the planet's orbit be the *xy* plane.

At $t = 0$ the planet's position is given by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and thirty days later ($t = 30$) it is at

$\begin{pmatrix} 0.500 \\ 0.866 \end{pmatrix}$

**a** Given that the planet's position at time *t* can be written as $\begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$ find the value of $\omega$

**b** Find the time for the planet to complete one orbit of the star.

At $t = 0$ a meteor is sighted at the point with coordinates $(1.12, 1.58, 2.02)$ travelling with velocity $\begin{pmatrix} -0.0260 \\ -0.0118 \\ -0.0400 \end{pmatrix}$. The *z* axis is taken as perpendicular to the plane of the planets orbit.

**c** Assuming the meteor continues with this trajectory, find

**i** an equation for the position of the meteor at time *t*

**ii** the value of *t* at which the meteor crosses the plane of the planet's orbit.

**d** **i** Find how far in astronomical units the meteor is from the planet when it crosses the plane of its orbit

Given the radius of the planet's orbit is 150,000,000km

**ii** find this distance in kilometres

**9** A parachutist of mass 60kg jumps from a plane flying at 450m above the ground. She falls under gravity and is subject to a resistance force proportional to her velocity. Let *t* be the time in seconds from when she began her fall.

Her motion can be modeled by the following second order differential equation

$\ddot{x} = 9.8 - 0.2\dot{x}$

**a** **i** Use the substitution $\dot{x} = y$ and the technique of separating the variables to show that $\dot{x} = 49\left(1 - e^{-0.2t}\right)$

**ii** Find the velocity of the parachutist when $t = 5$

**iii** By integrating the equation found in part i find the distance she has fallen when the parachute is opened.

She opens her parachute after 5 seconds. Her subsequent motion can be described by the following differential equation.

$\ddot{x} = 9.8 - 0.5\dot{x}^2$

**b** Using the substitution $\dot{x} = y$ and Euler's method with a step length of 0.2, find the time at which she hits the ground and her velocity at this time.

**Exam-style questions**

**10** On a particular area of ice in the Artic there is a population of seals and a population of polar bears. The seals are hunted by the polar bears. Let $x$ represent the number of seals measured in thousands, and let $y$ represent the number of polar bears measured in hundreds. Let $t$ represent time in years. Initially there are $S$ thousand seals and $P$ hundred polar bears. The situation is modelled by the coupled differential equations

$$\begin{aligned}\dot{x} &= 2x - axy \\ \dot{y} &= xy - by\end{aligned}.$$

The two equilibrium points for this model are $x = 0, y = 0$ or $x = 10, y = 2$.

**a** Find the constants $a$ and $b$. (3)

**b** If initially the number of seals and the number of polar bears were both increasing write down an inequality for $S$ and an inequality for $P$. (3)

**c** If initially the number of seals were decreasing but the number of polar bears were increasing write down an inequality for $S$ and an inequality for $P$. (3)

**11** Two species of fungi $F$ and $G$ are symbiotic; each helps the other to prosper.

In a particular area, let $x$ be the population of species $F$ and $y$ the population of species $G$, both measured in hundreds. Time $t$ will be measure in years. The situation is modelled by the two coupled differential equations

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x\end{aligned}, \text{ where initially } x = 5 \text{ and } y = 3.$$

**a** Use the matrix-eigenvalue method to find the solution to these two differential equations. (10)

**b** Find $\lim_{t\to\infty} \frac{x}{y}$ and comment on the meaning of this in the context of the situation. (2)

**12** When a force $\mathbf{F}$ moves an object through a displacement of $\mathbf{d}$ the work done is given by the scalar $\mathbf{F} \cdot \mathbf{d}$.

**a** Find the work done when the force $\mathbf{F} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ moves an object through a displacement of $\mathbf{d} = \begin{pmatrix}4\\5\\6\end{pmatrix}$. (2)

**b** Let $\mathbf{F} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, $\mathbf{G} = \begin{pmatrix}d\\e\\f\end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. Prove that $\mathbf{F} \cdot \mathbf{d} + \mathbf{G} \cdot \mathbf{d} = (\mathbf{F} + \mathbf{G}) \cdot \mathbf{d}$. (3)

When a force $\mathbf{F}$ is applied at a point $P$ with position vector of $\overrightarrow{\mathbf{OP}}$ relative to the origin $O$, then $\mathbf{F} \times \overrightarrow{\mathbf{OP}}$ represents the turning effect (torque) about the origin. $\mathbf{F} \times \overrightarrow{\mathbf{OP}}$ will be a vector perpendicular to the plane of the rotation.

**c** Find the turning effect when $\mathbf{F} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ and $P = (3, 4, 0)$. (2)

**13** An object acting under gravity has an acceleration vector of $\mathbf{a} = \begin{pmatrix}0\\-10\end{pmatrix}$ m s$^{-2}$.

A stone is thrown at time $t = 0$ from the origin with speed of $10$ m s$^{-2}$ at an angle of $\theta$ ($\arctan\frac{1}{2} < \theta < 90^\circ$) to the positive horizontal $x$ axis.

**a** Show that its velocity vector is given by $\mathbf{v} = \begin{pmatrix} 10\cos\theta \\ 10\sin\theta - 10t \end{pmatrix}$ . (2)

**b** Find the displacement vector $\mathbf{s}$. (2)

The stone is thrown to land on a slope through the origin which is parallel to the vector $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

**c** Find the scalar component of $\mathbf{s}$ perpendicular to $\mathbf{w}$. (2)

**d** Hence find the time (other than $t = 0$) when this perpendicular component is zero. (2)

**Answers**

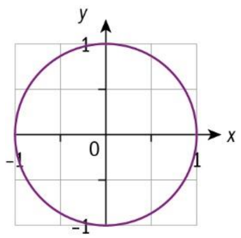
**1** **a** $\begin{pmatrix} 4\cos(t^2) - 8t^2\sin(t^2) \\ 18\cos(3t) \end{pmatrix}$

**b** $\begin{pmatrix} 2\sin(t^2) + 4 \\ -2\cos(3t) + t + 2 \end{pmatrix}$

**2** **a** **i** $\pm 2\text{i}$

**ii** $\dfrac{dx}{dt} = 0,\ \dfrac{dy}{dt} = 2$

**iii**

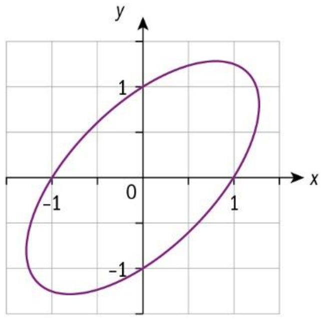



**iv** A circle centre (0,0)

**b** **i** $\pm 2\text{i}$

**ii** $\dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = 1$

**iii**




**iv** An ellipse centre (0,0)

**3** $a = 1,\ b = -1$

**4** 1.22 m

**5** **a** $\begin{pmatrix} 1.5 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -0.616 \\ 1.69 \end{pmatrix}$

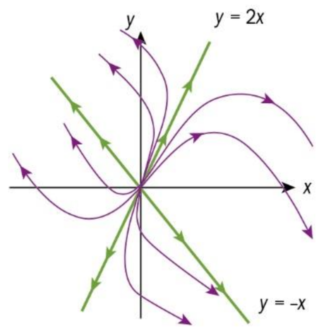
**b** **i** 0.884 $\text{ms}^{-1}$ **ii** 1.69 $\text{ms}^{-1}$

**c** **i** 11.8 sec **ii** 10.5 m downstream.

**6** **a** 0.5 and 0.2

**b** $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{0.5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{0.2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

**c**




**d** $y_0 > 2x_0$

**7** **a** $\begin{pmatrix} d \\ h - \frac{1}{2}gt^2 \end{pmatrix}$

**b** $\begin{pmatrix} V\cos(\alpha)t \\ V\sin(\alpha)t - \frac{1}{2}gt^2 \end{pmatrix}$

**c** Show that $\tan\alpha = \dfrac{h}{d}$

**8** **a** $0.0349\left(\dfrac{\pi}{90}\right)$

**b** 180 days

**c** **i** $\boldsymbol{r} = \begin{pmatrix} 1.12 \\ 1.58 \\ 2.02 \end{pmatrix} + t\begin{pmatrix} -0.0260 \\ -0.0118 \\ -0.0400 \end{pmatrix}$

**ii** $t = 50.5$ days

**d** **i** 0.0033 astronomical units.

**ii** 495,000km

**9** **a** **ii** 19.3ms$^{-1}$

**iii** 335m

**b** 21.2 seconds, 7ms$^{-1}$

**10 a** $\dot{x} = x(2 - ay) = 0, \dot{y} = y(x - b) = 0 \Rightarrow x = 0, y = 0$ or $x = b, y = \dfrac{2}{a}$

So $b = 10, a = 1$ M1A1A1

**b** $\dot{x} > 0, \dot{y} > 0 \Rightarrow 2 - B > 0,\ B < 2,$ and $S - 10 > 0,\ S > 10$ R1A1A1

**c** $\dot{x} < 0, \dot{y} > 0 \Rightarrow 2 - B < 0,\ B > 2,$ and $S - 10 > 0,\ S > 10$ R1A1A1

**11 a** Matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1 \text{ or } -1$ M1A1A1

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 1\begin{pmatrix} p \\ q \end{pmatrix}$ gives $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as an eigen-vector A1

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = -1\begin{pmatrix} p \\ q \end{pmatrix}$ gives $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as an eigen-vector A1

Solution is $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ A1

Initial conditions give $A + B = 5, A - B = 3 \Rightarrow A = 4, B = 1$ . Hence M1A1

$x = 4e^{t} + e^{-t} \qquad y = 4e^{t} - e^{-t}$ A1A1

**b** $\lim\limits_{t\to\infty} \frac{x}{y} = 1$, so the two population numbers approach each other as time increases. A1R1

**12 a** $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 32$ M1A1

**b** LHS $= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz + dx + ey + fz$ M1A1

RHS $= \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (a+d)x + (b+e)y + (c+f)z$ =LHS completing the proof A1

**c** $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$ M1A1

**13 a** Integrating $\mathbf{a} \Rightarrow \mathbf{v} = \begin{pmatrix} 0 \\ -10t \end{pmatrix} + \mathbf{c}$, and initial conditions give $\mathbf{c} = \begin{pmatrix} 10\cos\theta \\ 10\sin\theta \end{pmatrix}$ and hence the desired result. M1A1

**b** Integrating again $\mathbf{d} = \begin{pmatrix} 10t\cos\theta \\ 10t\sin\theta - 5t^2 \end{pmatrix}$ since starts at the origin. M1A1

**c** $\dfrac{|\mathbf{d} \times \mathbf{w}|}{|\mathbf{d}|} = \dfrac{|10t\cos\theta - 20t\sin\theta + 10t^2|}{\sqrt{5}}$ M1A1

**d** $|\mathbf{d} \times \mathbf{w}| = 0 \Rightarrow 10\cos\theta - 20\sin\theta + 10t = 0 \Rightarrow t = 2\sin\theta - \cos\theta$ M1A1

# 13 Representing multiple outcomes: random variables and probability distributions

**1** A decahedral die with faces numbered 1,2,3,…,10 is repeatedly thrown until an 8 is obtained. Let $D$ be the number of throws carried out.

**a** Find the probability distribution function $f(d) = P(D = d)$ and state its domain.

**b** What is the most likely number of rolls?

**c** Use technology to help you estimate $E(D)$.

**d** Find $P(D \geq 5)$.

**2** In the manufacturing process of patterned silk by Zig Fabrics, imperfections occur randomly. The number of imperfections can be modelled by a Poisson distribution with mean 0.9 faults per $10\,m^2$ . David is deciding if he should order patterned silk from Zig Fabrics in order to make silk scarves of dimensions 20 cm by 140 cm.

**a** Find the probability that a randomly chosen scarf made from Zig Fabrics has at least one imperfection.

**b** In a batch of 5000 scarves, find how many David would expect to have at least two imperfections.

**c** Each perfect scarf sold makes David a profit of $10.00. Scarves with one imperfection are sold at a discount, making David a profit of $2. Scarves with two or more imperfections are sold for no profit.

Calculate David's expected profit from selling 15000 scarves.

**3** A 10 km challenge walk is organized to raise funds for a charity. Data collected from previous years shows that the finishing times follow a normal distribution with mean 100 minutes and standard deviation 25 minutes.

**a** Find the probability that a randomly chosen participant will finish the walk in less than two hours.

**b** The fastest 10% of the participants receive a certificate of commendation. Find to the nearest minute the time below which a participant must finish the walk, in order to gain a certificate.

**c** Zeke organizes a team of five to take part in the walk together. The team of five all finish the walk. Find the probability that at least two of the team finish the walk in under 110 minutes.

**4** In Chancer's café, the manager notices that customers either pay by contactless payment or by cash. After examining her receipts over a period of 6 months she estimates that 83% of customers pay with contactless. All other payments are in cash.

Assume the payment method chosen by each customer is independent of that chosen by any other customer and that the average time taken to process a contactless payment is 5 seconds and a cash payment 15 seconds.

**a** Find the expected processing time of a randomly chosen customer's payment.

**b** In a queue of eight people, find the probability that the first five customers all pay with cash and the rest with contactless.

**c** Find the probability that at least five of the eight customers pay with cash.

**5** Anrai and Clara design a game in which two tetrahedral die with faces numbered 1,2,3 and 4 are thrown. $S$ is the total of the two numbers thrown on the dice.

**a** Represent the probability distribution of $S$ in a table.

**b** Represent the probability distribution of $S$ as a function $g(s) = P(S = s)$.

**c** Anrai and Clara's game costs $\$x$ to play. Players win $\$(x+1)$ if $S$ is a multiple of 4 and if $S$ is 6 they win $\$(x+2)$. Otherwise, they lose the $\$x$ they paid to play. Find the value of $\$x$ that makes the game fair.

**6** Bags of coffee beans from a plantation are produced independently in regular or large sizes. The weights of each type of bag $R$ and $L$ are distributed normally as shown in the table:

| | Mean (g) | Standard deviation (g) |
|---|---|---|
| $R$ | 512 | 10 |
| $L$ | 2027 | 32 |

**a** Tomas buys four large bags and four regular bags. Find the probability that the total weight of coffee beans is less than 10 kg.

**b** Tomas selects one of his large bags and one of his regular bags at random. Find the probability that the large bag weighs more than four times the weight of the regular bag.

**7** A company employs two consultants, Francesca and Jakub, to draft a new website. It is known that Francesca makes on average 2.2 typographical errors per 10 MB of website content and Jakub makes on average 3.8 typographical errors per 9 MB of website content. In one month, working independently, Francesca delivers 78 MB of content and Jakub 56 MB. Assuming the number of errors follow a Poisson distribution, find the probability that together Francesca and Jakub make 45 or more errors between them.

**8** Car hire company RapidCar buys batteries for its fleet from a supplier who states that the life of the batteries it manufactures, $L$, has mean 5 years and standard deviation 1 year. RapidCar wants to investigate this claim by examining a sample of batteries from its fleet of cars. What sample size should be taken so that the management of RapidCar can be 99% certain that the mean of the sample is within 3 months of the population mean?

**9** Pietro the property developer is investigating where to invest money in real estate in a major city. In district A of the city, one-bed apartments have an average price of \$250,000 and a standard deviation of \$100,000. In district B, the mean price of a one-bed apartment is \$260,000 with a standard deviation of \$120,000. Pietro wishes to buy 40 apartments at random in district A and 35 in district B. In which district is it more likely that the average price of an apartment Pietro buys is greater than \$270,000?

**Exam-style questions**

**10** The probability distribution function for a discrete random variable, $X$, is given by

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X = x)$ | $a$ | $b$ | $c$ | $\frac{1}{7}$ |

The mean of $X$ is $\frac{18}{7}$ and its variance is $\frac{40}{49}$.

Find the value of the constants $a, b$ and $c$. (10)

**11** The discrete random variable, $X$, satisfies the $B(4, p)$ distribution, where $p \neq 0, p \neq 1$.
Given that $P(X = 1) = P(X = 2)$ find the value of $p$. (7)

**12** A discrete random variable, $X$, has mean of 1 and variance of 7. Another discrete random variable, $Y$, has mean of 5 and variance of 1.

**a** State (with a reason) whether either of these two random variables could satisfy a Poisson distribution. (2)

A third discrete random variable, $W$, is defined by $W = X + kY$ and it is given that $W$ *does* satisfy a Poisson distribution.

**b** Find the possible values for the constant $k$ and state what the Poisson distribution satisfied is, in each case. (6)

**13** Paul competes in Triathlons where he has to run 1.5 km, cycle 40 km and run 10 km. Let $S, B$ and $R$ be the random variables that represent his swim time, bike time and run time respectively.

It is known that $S \sim N(25, 1)$, $B \sim N(69, 9)$ and $R \sim N(35, 4)$ where the units are minutes. He is so fit that these three variables are independent of each other.

**a** Find the probability that in his next triathlon he fulfils his aim of the total time $T = S + B + R$ being less than 2 hours. (6)

**b** Find the probability that in his next triathlon his bike time is less that twice his run time. (6)

**Answers**

**1 a** $f(d) = \frac{1}{10} \times \left(\frac{9}{10}\right)^{d-1}$

**b** 1

**c** 10

**d** 0.6561

**2 a** 0.024885 **b** 1.56 scarves **c** \$147004

**3 a** 0.788 **b** 68 minutes **c** 0.949

**4 a** 6.7 s **b** 0.000081 **c** 0.00504

**5 a**

| $s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $P(S=s)$ | $\frac{1}{16}$ | $\frac{2}{16}$ | $\frac{3}{16}$ | $\frac{4}{16}$ | $\frac{3}{16}$ | $\frac{2}{16}$ | $\frac{1}{16}$ |

**b** $P(S=s) = \begin{cases} \frac{s-1}{16} & 2 \le s \le 5 \\ \frac{9-s}{16} & 6 \le s \le 8 \end{cases}$

**c** \$5

**6 a** 0.0100 **b** 0.565

**7** 0.2756

**8** 106

**9** District B

**10** $\sum P(X=x) = 1 \Rightarrow a+b+c+\frac{1}{7} = 1 \Rightarrow a+b+c = \frac{6}{7}$ M1A1

$\mu = \sum xP(X=x) \Rightarrow 1a+2b+3c+\frac{4}{7} = \frac{18}{7} \Rightarrow a+2b+3c = 2$ M1A1

$\sigma^2 = \sum x^2P(X=x) - \mu^2 \Rightarrow 1a+4b+9c+\frac{16}{7} - \frac{324}{49} = \frac{40}{49} \Rightarrow a+4b+9c = \frac{36}{7}$ M1A1

Solving 3 equations in 3 unknowns gives $a = \frac{1}{7}, b = \frac{2}{7}, c = \frac{3}{7}$ M1A1A1A1

**11** $4pq^3 = 6p^2q^2$ M1A1A1

$\Rightarrow 4q = 6p \Rightarrow 4(1-p) = 6p \Rightarrow 4 = 10p$ M1A1A1

So $p = \frac{2}{5}$ A1

**12 a** For the Poisson $\mu = \sigma^2$ so neither of these can be Poisson R1A1

**b** Mean of $W$ is $1+5k$ Variance of $W$ is $7+k^2$ A1A1

$1+5k = 7+k^2 \Rightarrow k^2 - 5k + 6 = 0 \Rightarrow k = 2 \text{ or } 3$ M1A1

$k = 2$ gives $P_0(11)$ $k = 3$ gives $P_0(16)$ A1A1

**13 a** $T$ satisfies the normal distribution with mean of $25 + 69 + 35 = 129$ and variance of $1 + 9 + 4 = 14$ R1M1A1A1

$P(T < 120) = 0.00808(3sf)$ M1A1

**b** Consider $W = 2R - B$ . $W$ satisfies the normal distribution with mean of $2 \times 35 - 69 = 1$ and variance of $2^2 \times 4 + 9 = 25$ M1A1M1A1

$P(W > 0) = 0.579(3sf)$ M1A1

# 14 Testing for validity: Spearman's hypothesis testing and $X^2$ test for independence

**1** It is thought that in a particular school the heights of 10th grade students can be modeled by a normal distribution.

The heights of 500 of the students are measured and the results shown in the table below.

| Height, $x$ cm | $144 \le x < 152$ | $152 \le x < 156$ | $156 \le x < 160$ | $160 \le x < 164$ | $164 \le x < 172$ |
|---|---|---|---|---|---|
| Frequency | 21 | 92 | 188 | 145 | 54 |

It is decided to use a chi-squared goodness of fit test to see if this hypothesis is justified.

**a** Calculate the mean and standard deviation of the sample

**b** State suitable null and alternative hypotheses.

**c** Explain why you need to use $x < 152$ and $x > 172$ as your end intervals for the calculation.

**d** Perform a goodness of fit test, clearly stating the degrees of freedom used, to see whether the data could be modelled by a normal distribution.

**2** Karl Pearson, one of the founders of the discipline of Statistics, wanted to see if there was a link between the types of crimes committed and the amount of alcohol the perpetrators drank. In 1909 he collected data, some of which is reproduced below.

| | Drinker | Abstainer |
|---|---|---|
| Arson | 50 | 43 |
| Violence | 155 | 110 |
| Stealing | 379 | 300 |
| Counterfeiting | 18 | 14 |
| Fraud | 63 | 144 |

**a** Carry out a suitable test to see if whether or not a person is a drinker or an abstainer is independent of the crimes committed.

**b** Comment on your conclusion.

**3** In a memory test the subject is asked to repeat a list of objects which they have are allowed to look at for one minute. Becky is trying to improve her memory and so performs the test several times and succeeds half the time. She then goes on a memory course and when she returns she takes the test five more times to see if her memory has improved.

**a** Write down the null and alternative hypothesis for this test.

**b** Under the null hypothesis write down the expected value and the variance of the number of successes Becky will obtain.

**c** Find the critical region for a test at the

**i** 5% **ii** 10% significance level.

**d** Becky achieves 4 successes out of the 5 tests. State the conclusion for the test, justifying your answer.

**4** The ages of 10 employees in a company and their salaries (in 1000s) are recorded in the table below.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Age ($x$) | 18 | 19 | 20 | 22 | 24 | 26 | 28 | 30 | 34 | 36 |
| Salary ($y$) | 34 | 38 | 39 | 40 | 41 | 43 | 43 | 45 | 44 | 48 |

**a** Perform a hypothesis test to see if, based on this data, the population correlation coefficient ( $\rho$ ) is equal to zero.

**b** If appropriate to do so find the equation of the line of regression in the form $y = ax + b$ .

**c** **i** Explain the significance of $a$

**ii** Explain why we cannot say the value of $b$ is the salary earned when $x$=0

**d** Use your line of regression to find the expected salary for a teacher aged 32.

**5** The long term average number of points achieved by Melchester United in a season follows a Poisson distribution with a mean of 62.

One year under a new manager the team achieved 71 points.

**a** test at the 5% level the hypothesis that the manager has improved the team.

The following season the chairman of the club decides to sack the manager if he underachieves. He regards the new mean as 71 points and will sack the manager if the total is less than 64 points.

**b** State the null and alternative hypothesis being used by the chairman and find the significance level of his test.

**c** If the team created by the manager would on average achieve 65 points a season find the probability the manager will be sacked.

**6** The number of patients per hour arriving at the emergency room of a hospital was determined over a 100 hour period from hospital records. The information gathered is given in the following table.

| | | | | | | |
|---|---|---|---|---|---|---|
| No. of patients | 12-14 | 15-17 | 18-20 | 21-23 | 24-26 | 27-29 |
| Frequency | 7 | 21 | 27 | 19 | 18 | 8 |

**a** Find the mean and standard deviation of the sample.

**b** State why you can assume the mean of the sample follows a normal distribution.

**c** Find a 95% confidence interval for the average number of patients arriving per hour at the emergency room.

**d** Test at the 5% level of significance whether the average number of patients arriving at the emergency room per hour ( $\mu$ ), is equal to 21.

**7** The employees in a company have been instructed not to lift weights above 25kg. Part of the job involves moving boxes and packages and one of the employees, Tom, normally carries 2 boxes and 1 package. His supervisor, Emily, is concerned, so she does some checks on the weights of the boxes and packages.

She collects five boxes and 10 packages and weighs them. The boxes have a mean weight of 7.5kg and the packages have a mean weight of 3kg. The weights can be assumed to come from a normal distribution and standard deviation of boxes is known to be 1.5kg and that of the packages is 0.8 kg.

**a** Calculate a 95% confidence interval for the weights of the boxes and the packages.

Emily decides she will use the upper boundary of the confidence intervals as an estimate for the mean weight of the boxes and packages to see how often Tom will carry a load greater than 25kg.

She decides that if the probability of a load being greater than 25kg when using these values is less than 0.05 she will allow him to continue.

**b** Use these figures to find the probability the weight of a load will be greater than 25kg and hence whether Tom can continue to carry two boxes and one package.

**8** Alec is playing a game which involves rolling a six sided dice and a four sided dice. The six sided dice has the numbers 1 to 6 on its faces and the four sided dice has the numbers 2, 2, 4, 6. The total is the sum of the uppermost face on the six-sided dice and the face that is flat on the surface of the table for the four-sided dice.

**a** Write down the probability the total is an odd number.

**b** Find the probability the total is 8

**c** Find

**i** the expected total score

**ii** the variance of the total score.

The dice are rolled 100 times and the scores noted. From looking at the data Alec believes that at least one of the one of the dice must be biased.

**d** Test this belief at the 5% level given that the total score from the 100 rolls is 655.

**9** The average number of houses sold by each an agency follows a Poisson distribution with a mean of 1.8 per day. A new team takes over one branch and the number of houses sold ($X$) over a 10 day period is noted.

**a** Find the total number of houses the branch will need to sell during the 10 day period to convince the agency managers that they have improved the average at the 5% significance level and hence state the value of a type I error.

**b** Write down the average number of houses per day the branch would have to sell for the result to be significant at the 5% level.

**c** If strategies the branch have put in place will improve their average number of sales per day to 2.1 find the probability of a type II error.

The number of days included in the sample is now increased to 40.

**d** Find the total number of houses, and hence the average number of houses per day, the branch will need to sell to convince the agency managers that they have improved the average at the 5% significance level.

**e** From a consideration of $Var\left(\frac{X}{n}\right)$, where $n$ is the number of days over which the sample is taken, explain why your answers to b and d are different.

**Exam-style questions**

**10** Yoghurts of mixed flavours come in packs of four. It is believed that the probability of a yoghurt being strawberry flavour is $\frac{1}{3}$. Let the discrete random variable, $X$, be the number of strawberry flavoured yoghurts that are found in a pack. The null hypothesis is that $X$ satisfies the $B\left(4, \frac{1}{3}\right)$ distribution. To test this, a sample of 100 packs (each containing four yoghurts) are investigated and the results given in the table below.

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Observed frequency | 22 | 38 | 26 | 13 | 1 |

**a** Construct a similar table to the one above to show the expected frequencies (to two decimal places) under the null hypothesis. (2)

**b** By considering the conditions for a suitable test to be valid, construct a modified table giving the observed and expected frequencies, giving a reason for the decision. (3)

**c** Test the null hypothesis at the 10% significance level. Your answer should include: the test used, the number of degrees of freedom (with a reason), the $p$ value and the conclusion of the test (with a reason). (7)

**11** The manufacturers claim that 4 out of 5 cats prefer eating their product *Mousemeat* to any other cat food. It is suspected that they are exaggerating. A sample of 15 cats was taken and it was discovered that only 9 of them preferred *Mousemeat.* It can be assumed that all cats act independently of each other.

**a** Test at the 10% level whether this is sufficient evidence to conclude that the manufacturers are exaggerating. In your answer, you should: state if this is a one tailed or two tailed test, write down the test hypotheses, state the distribution used, calculate the $p$ value and write down the conclusion (with a reason) of the test. (8)

**b** State if the conclusion would have been any different if working at the 5% level. (2)

**12** There are two species of fish, Species L and Species H, which are very difficult to distinguish between. These fish swim in shoals made up of entirely one species of fish.

The mass of Species L is normally distributed with mean of 1 kg and standard deviation of 0.2 kg. The mass of the heavier species, Species H, is normally distributed with mean of 1.2 kg and standard deviation of 0.3 kg.

Sarah, a marine biologist, has caught 16 of these fish from a single shoal. She wishes to try and determine which species they belong to by examining their masses. She will find the sample mean mass $\bar{x}$ of the 16 fish. She has made up the following decision rules:

If $\bar{x} \leq 1.1$ she will accept the null hypothesis, that the fish belong to Species L.

If $\bar{x} > 1.1$ she will accept the alternative hypothesis, that the fish belong to Species H.

**a** Find the probability that she makes a Type I error. (5)

**b** Find the probability that she makes a Type II error. (5)

**13** A manufacturer of breakfast cereal produces packages that are nominally of mass 500 g. A trading standards officer is suspicious that the manufacturer is selling underweight packages. He takes a sample of 10 packages and discovers that the sample mean is $\bar{x} = 495g$ and that the sample standard deviation is $s_n = 5$.

**a** Calculate an estimate for the population standard deviation. (2)

**b** Carry out a test, at the 5% significance level, to determine if the manufacturer is selling underweight packages. Your answer should include the test used (with a reason), the test hypotheses, the $p$ value and (with a reason) the conclusion of the test. (8)

**Answers**

**1 a** $\bar{x} = 159\text{ cm}$, $s_n = 4.63$, $(s_{n-1} = 4.63)$

**b** $H_0$: The data is from a $N(159, 4.63^2)$ population.

$H_1$: The data is not from a $N(159, 4.63^2)$ population.

**c** Because the expected values for the region below 144 cm or above 172 cm might not be zero.

**d** Degrees of freedom = 2, $p$-value $= 0.00233$

Very strong evidence that the data is not from a normal population.

**2 a** $H_0$: drinking and the type of crime committed are independent.

$H_1$: drinking and the type of crime committed are not independent.

$p$-value = $1.29 \times 10^{-9}$

$1.29 \times 10^{-9} < 0.01$ the result is strongly significant and $H_0$ can be rejected.

**b** The significant feature is that all the categories except fraud have slightly more drinkers than abstainers. Fraud is very strongly the other way and this is the reason the null hypothesis is rejected.

**3 a** $H_0$: $p = 0.5$

$H_1$: $p > 0.5$

**b** 2.5, 1.25

**c i** $X = 5$ **ii** $X \geq 4$

**d** $X$ is only in the critical region for the 10% significance level so the alternative hypothesis that Becky has improved is only accepted at the 10% level.

**4 a** $H_0$: $\rho = 0$

$H_1$: $p \neq 0$

$p$-value = $9.21 \times 10^{-5}$ < 0.01, significant so strong evidence to reject $H_0$ that the population correlation coefficient is equal to zero.

**b** $y = 0.592x + 26.3$

**c i** Within the given range each year of age on average adds $592 onto the salary of an employee.

**ii** the value of $y$ at $x$=0 would be salary at age 0. You should not extrapolate the line outside the domain of the data given.

**d** $45200

**5 a** $H_0$: $\mu = 62$

$H_1$: $\mu > 62$

$p$-value = 0.141>0.05, not significant so insufficient evidence at the 5% level of improvement in the long term average.

**b** $H_0$: $\mu = 71$

$H_1$: $\mu < 71$

significance level = 0.188 or 18.8%

**c** 0.434

**6 a** $\bar{x} = 20.32,\ s_n = 4.16$

**b** The sample is large enough for the central limit theorem to apply.

**c** $(19.49, 21.15)$

**d** $H_0$: $\mu = 21$

$H_1$: $\mu \neq 21$

$p$-value = 0.107, not significant at the 5% level, so no reason to reject $H_0$.

**7 a** Boxes $(6.185, 8.815)$ Packets $(2.504, 3.496)$

**b** Let $W$ equal the total weight

$W \sim N(21.125, 5.14)$

$P(W > 25) = 0.0437$

Tom is allowed to continue carrying two boxes and one package.

**8 a** $\frac{1}{2}$

**b** $\frac{1}{6} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{4} = \frac{1}{8}$

**c** Let $T$ be the total score on a single roll

**i** $E(T) = 7$ **ii** $Var(T) = 5.67$

**d** $\bar{t} = 6.55 \quad Var(\bar{T}) = \frac{5.67}{100}$

$H_0$: $\mu = 7$

$H_1$: $\mu \neq 7$

Under $H_0$, by the central limit theorem $\bar{T} \sim N\left(7, \frac{5.67}{100}\right)$

$p$-value = $2 \times 0.0294 = 0.0588$ > 0.05

or critical region is $\bar{T} < 6.53,\ \bar{T} > 7.47$,

6.55>6.53

so not significant at 5% so insufficient evidence to reject $H_0$.

**9 a** $P(X \geq a) \leq 0.05$

$P(X \leq a-1) \geq 0.95 \Rightarrow a-1 = 25 \quad \Rightarrow a = 26$

Critical region $X \geq 26$

Probability of a type I error = 0.04461

**b** 2.6

**c** $P(X \leq 25 \mid \mu = 21) = 0.838$

**d** Critical region $X \geq 87$ so $\bar{X} \geq 2.175$

**e** $Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X) = \frac{1.8n}{n^2} = \frac{1.8}{n}$. Variance decreases as $n$ increases, so there is likely to be less variability in the sample and the critical value will be closer to the value assumed under $H_0$.

**10 a**

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Expected frequency | 19.75 | 39.51 | 29.63 | 9.88 | 1.23 |

A2

**b** Require the expected value to be greater than 5 so have to combine cells. R1

| $X$ | 0 | 1 | 2 | $\geq 3$ |
|---|---|---|---|---|
| Expected frequency | 19.75 | 39.51 | 29.63 | 11.11 |
| Observed frequencies | 22 | 38 | 26 | 14 |

M1A1

**c** $\chi^2$ goodness of fit test A1

$H_0$ : observed data fits the $B(4, \frac{1}{3})$ distribution $H_1$ : it does not

Degrees of freedom =3. 4 cells -1 restriction (totals) A1R1

the $p$ value=0.680 A2

since $0.680 > 0.1$ we accept the null hypothesis that $X$ satisfies the $B(4, \frac{1}{3})$ distribution.

R1A1

**11 a** 1 tailed test A1

$H_0$ : probability of preferring *Mousemeat* is $\frac{4}{5}$. $H_1$ : it is less than this. A1A1

Under $H_0$, distribution is $B\left(15, \frac{4}{5}\right)$ A1

$p$ value $= P(X \leq 9) = 0.0611$ M1A1

$0.0611 < 0.10$ so the null hypothesis is rejected and it is concluded that the manufacturers are exaggerating. R1A1

**b** $0.0611 > 0.05$ so at the 5% level there is not enough evidence to disbelieve the manufacturers. R1A1

**12 a** $P(\text{Type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true})$ Under $H_0$ $\bar{X} \sim N\left(1, \frac{0.2}{4}\right)$ R1M1A1

$P\left(\bar{X} > 1.1\right) = 0.0228(3sf)$ M1A1

**b** $P(\text{Type II error}) = P\left(\text{accept } H_0 \,|\, H_1 \text{ is true}\right)$ Under $H_1$ $\bar{X} \sim N\left(1.2, \dfrac{0.3}{4}\right)$ R1M1A1

$P\left(\bar{X} \le 1.1\right) = 0.0912(3sf)$ M1A1

**13 a** Estimate is $s_{n-1} = \sqrt{\dfrac{10}{9}} \times 5 = 5.27(3sf)$ M1A1

**b** $t$ test as the population variance is unknown A1R1

$H_0 : \mu = 500$, $H_1 : \mu < 500$ A1A1

p value $= 0.00748$ A2

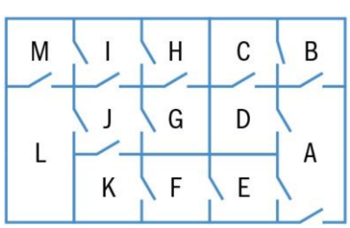
$0.00748 < 0.05$ so we reject $H_0$ and conclude that the (naughty) manufacturer is selling underweight packages. R1A1

# 15 Optimizing complex networks: graph theory

**1** Eight villages need to be connected by water pipes. The distances between the villages in kilometers are shown in the table below. Use Prim's algorithm, beginning with vertex P, to find the minimum spanning tree and hence the minimum cost of connecting all the villages.

| | P | Q | R | S | T | U | V | W |
|---|---|---|---|---|---|---|---|---|
| P | 0 | 4 | 12 | 13 | 8 | 11 | 14 | 10 |
| Q | 4 | 0 | 7 | 10 | 11 | 13 | 10 | 3 |
| R | 12 | 7 | 0 | 5 | 9 | 6 | 4 | 8 |
| S | 13 | 10 | 5 | 0 | 4 | 6 | 9 | 13 |
| T | 8 | 11 | 9 | 4 | 0 | 8 | 12 | 10 |
| U | 11 | 13 | 6 | 6 | 8 | 0 | 4 | 11 |
| V | 14 | 10 | 4 | 9 | 12 | 4 | 0 | 12 |
| W | 10 | 3 | 8 | 13 | 10 | 11 | 12 | 0 |

**2** A security guard locks up a stately home every evening in the following way. He walks through each of the internal doors and then locks the door behind him. A plan of the stately home is shown below.



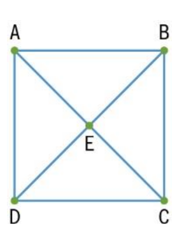

He needs to finish his walk in room A so he can leave the building.

**a** Show the plan of the of the stately home as a graph with appropriate vertices and edges.

**b** In which room will he need to begin his walk

**c** Give a possible route he could take to lock all the doors.

**3** A sewing machine needs sew on five buttons onto a tunic as shown.



A, B, C and D are in the shape of a square of side length 10 cm and E is in the centre of the square. Initially the sewing machine needs to position itself over point A and to return to point A when all the buttons have been sewn on.

**a** State whether the route taken by the sewing machine should be a Hamiltonian cycle or a Eulerian circuit. Justify your answer.

**b** List all possible routes of minimum length if the sewing machine begins by sewing the buttons on at points A, B and C and ends at point A

**c** Find the total length moved by the sewing machine when sewing on the buttons.

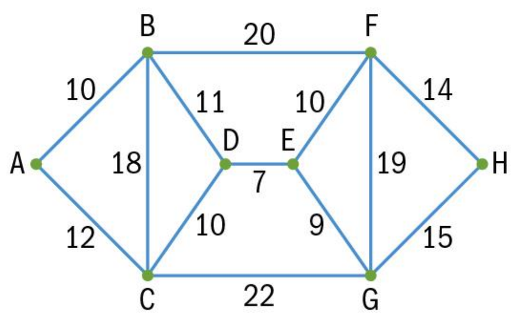
In order to save time it is decided that the sewing machine does not need to return to position A. So if it finishes at position B then it can begin the next tunic also at position B.

**d** **i** Which of the routes listed in B would you recommend to the

manufacturer and at what point would it finish the tunic?

**ii** What would be the length of movement of the sewing machine saved per tunic?

**4** A road inspector is to check all the roads shown in the graph below. The weights on the edges are the times in minutes it will take him to check the roads.

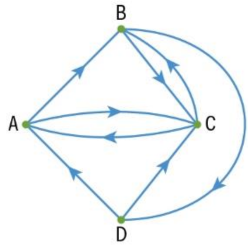



**a** Find the least amount of time it will take to check all the roads if he begins and ends the inspection at point A.

It is noticed that the roads connecting points B and D and B and F are twice as wide as the other roads so the inspector will need to walk both ways along them.

**b** Find how much longer will the inspection now take.

**5** A small network of webpages is shown by the directed graph below. The edges indicate the links between the webpages.



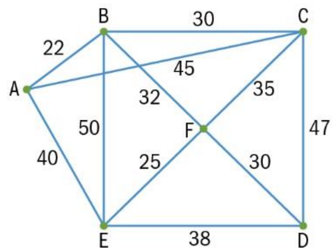

**a** Write down the transition matrix for this graph

**b** Write down a system of linear equations that can be solved to find the exact proportion of time a random walk would spend at each of the vertices.

**c** **i** Solve the system of equations

**ii** hence give the order in which the webpages in the network could be ranked by a search engine.

**6** The following diagram represents the cost (in \$1000) of connecting the offices of a company to a secure network.

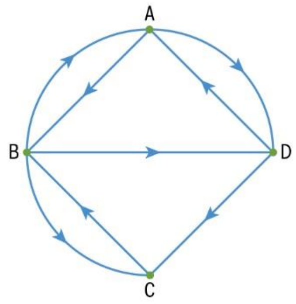



**a** Use Kruskal's algorithm to find the minimum cost of connecting the office. State the order in which the edges are selected and the total cost of the connections.

A few years after completion of the connections it is decide that to make the system more robust A, B and C must be connected in a circuit (which could also include other offices) so that if one of the links fails they will still be connected.

**b** Assuming the costs are still the same and any failures will occur in the links and not in the offices which extra connections would you recommend adding?

**7** Arnold (A) is connected in a network with three other people; Bertha (B), Charles (C) and Diana (D) as shown in the directed graph below.




He decides he will randomly click on three links in the network and see which friend he connects to.

**a** Construct an adjacency matrix for this network

Arnold decides he will work out the probability of being connected to Charles after three links by evaluating the number of walks of length three from his page to Charles' page and dividing this by the total number of walks of length three.

**b** By considering appropriate powers of the adjacency matrix calculate the probability Arnold obtains.

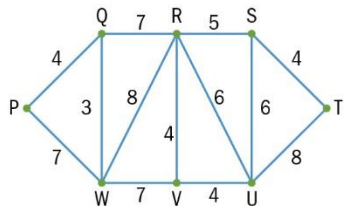
Arnold explains his method to Bertha who says he has calculated incorrectly and instead he should find the transition matrix for the graph and use this to calculate the probability of being connected to Charles after three slinks.

**c** Calculated the probability using Bertha's method.

**d** Explain why Arnold's method does not give the correct answer

**e** Justify the answer obtained in c with reference to the original graph.

**8** The graph shown shows the lengths of time it takes to walk the paths in a park. An inspector needs to walk along all the paths at least once and return to his starting point.




**a** Find which paths need to be repeated if his walk is to be of minimum length.

**b** It is now possible for the inspector to start and finish his walk at two different vertices. State which vertices he should start and finish at and list any paths that need to be repeated.

On a different day the inspector needs to visit each of the sites P to W and return to his starting point at P.

**c** Find the values of a, b, c and d in the table of least distances

| | P | Q | R | S | T | U | V | W |
|---|---|---|---|---|---|---|---|---|
| P | 0 | 4 | 11 | 16 | 20 | **a** | 14 | 7 |
| Q | 4 | 0 | 7 | 12 | 16 | **b** | 10 | 3 |
| R | 11 | 7 | 0 | 5 | 9 | 6 | 4 | 8 |
| S | 16 | 12 | 5 | 0 | 4 | 6 | **c** | 13 |
| T | 20 | 16 | 9 | 4 | 0 | 8 | 12 | **d** |
| U | **a** | **b** | 6 | 6 | 8 | 0 | 4 | 11 |
| V | 14 | 10 | 4 | **c** | 12 | 4 | 0 | 7 |
| W | 7 | 3 | 8 | 13 | **d** | 11 | 7 | 0 |

**d** Beginning at P the nearest neighbor algorithm leads to three possible upper bounds for his route around all the sites.

**i** State the total weight of each of these routes.

**ii** State the best upper bound

**iii** List the sites passed through by the inspector along this route.

The inspector decides to find a lower bound by using the deleted vertex algorithm and deleting vertex P

**e** Use Prim's algorithm on the table of least distances to find the weight of the minimum spanning tree of the subgraph formed by deleting P.

**f** Hence find a lower bound for the minimum length of time taken to visit all the sites.

**9** The adjacency matrix, **M** , represents links between webpages. For example the 1 in the A row and B column indicates there is a link from A to B.

$$\begin{array}{c|c} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

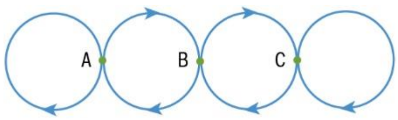
**a** Is it possible to move between any two of the webpages using fewer than four links? If not state the webpages for which this is not possible.

**b** Is it possible to move between any two of webpages in fewer than five links? Justify your answer.

**c** Is it possible to move between any two of the webpages in exactly four links? Justify your answer.

**Exam-style questions**

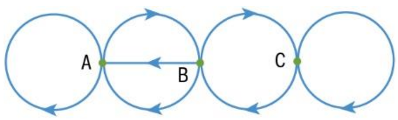
**10** This graph represents a random walk.




**a** Construct the corresponding transition matrix for this graph. (1)

**b** Find the steady state probability. (1)

This is now changed by adding an extra link.




**c** Construct the corresponding transition matrix for this graph. (1)

**d** Find the steady state probability by solving a linear system of equations. (2)

**e** Find the probability of the route *ABABC* if the starting state is $\left\{\frac{7}{10}, \frac{1}{10}, \frac{2}{10}\right\}$. (1)

**11** An airline wants to optimize its route system. The following table shows monthly operation costs of maintaining a route between the cities in thousands of pounds.

**a** Find the lowest cost to maintain this set of routes. (3)

The cities F and G must have a direct route, and cities A and B are not connected.

**b** Find the lowest cost to maintain such set of routes. (3)

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| A | 0 | 4 | 16 | 32 | 10 | 11 | 29 |
| B | 4 | 0 | 23 | 10 | 25 | 18 | 20 |
| C | 16 | 23 | 0 | 12 | 14 | 3 | 30 |
| D | 32 | 10 | 12 | 0 | 19 | 13 | 15 |
| E | 10 | 25 | 14 | 19 | 0 | 40 | 43 |
| F | 11 | 18 | 3 | 13 | 40 | 0 | 32 |
| G | 29 | 20 | 30 | 15 | 43 | 32 | 0 |

**12** The integral of $\frac{1}{x^2}$ between 1 and infinity is approximated by rectangles, where the function is evaluated at points $1, 1+\frac{1}{n}, \left(1+\frac{1}{n}\right)^2, \left(1+\frac{1}{n}\right)^3, \ldots$

**a** Find the width of the $k$-th rectangle. (1)

**b** Write the integral as an infinite sum and hence find the integral. (5)

**c** Perform the integral directly and confirm your answer. (2)

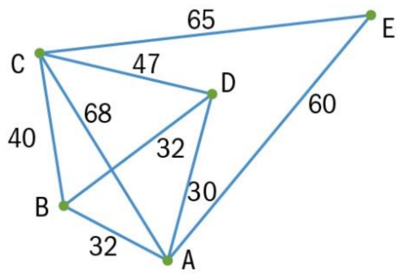
**13** The motion of a projectile affected by atmospheric resistive forces and gravitational field can be modelled as $\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt}$, $\frac{d^2y}{dt^2} = -\beta\frac{dy}{dt} - 9.8$, where $\alpha$ and $\beta$ are coefficients that depend on the strength of the resistive forces.

**a** Find the horizontal and vertical coordinates as functions of time, $x(t)$ and $y(t)$, if at $t = 0$ then $\frac{dx}{dt} = v_{x0}$, $\frac{dy}{dt} = v_{y0}$ and $x = 0, y = 0$. (6)

**b** Suppose $\alpha = \beta$. Write your answer from part **a** in the form $y(t) = a\left(1 - e^{-\beta t}\right) - bt$, and hence find the trajectory equation $y(x)$, which does not contain $t$. (5)

**c** Find the coordinates $(x, y)$ where the projectile reaches its maximum height, if $v_{x0} = v_{y0} = 10$ and $\beta = 0.01$. (5)

**14** The following graph represents a network of roads connecting five towns. The travelling salesman can choose which town to leave from.



**a** Find the upper bound for the travelling salesman problem, using the nearest neighbor algorithm. Start at different vertices to find the best possible upper bound. (7)

**b** Find the best lowest bound by removing each vertex from the graph. (7)

**c** Find the true solution to the travelling salesman problem starting from vertices A and D. Comment on your results. (3)

**Answers**

**1** Total distance is 31 km

**2** **b** I

**c** for example, IMLJIHGJKFEABCDA

**3** **a** A Hamiltonian cycle as it needs to visit all the vertices but not go along all the edges.

**b** ABCEDA, ABCDEA

**c** 44.1 cm

**d** **i** ABCED, finishing at D.

**ii** 10 cm

**4** **a** 177+7 = 184 minutes

**b** 177 + 20 + 11 +10 = 218

34 minutes

**5** **a**
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 \end{pmatrix} \end{array}$$

**b** Any three of

$c + d = 2a$
$a + c = 2b$
$a + b + d = 2c$
$b = 2d$

and $a + b + c + d = 1$ where *a, b, c, d* are the proportions of time spent at A,B,C and D

**c** **i** $a = \frac{5}{21}, b = \frac{2}{7}, c = \frac{1}{3}, d = \frac{1}{7}$

**ii** C, B, A, D

**6** **a** AB, EF, FD (BC), BC (FD), BF Total cost = $139,000

**b** Add CF and ED

**7 a**
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 0 & 1 \\ B & 1 & 0 & 1 & 1 \\ C & 0 & 1 & 0 & 0 \\ D & 1 & 0 & 1 & 0 \end{array}$$

**b**
$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 1 & 4 & 1 & 2 \\ B & 3 & 2 & 3 & 3 \\ C & 1 & 2 & 1 & 1 \\ D & 3 & 0 & 3 & 2 \end{array} \quad , \quad \frac{1}{8}$$

**c** $\frac{1}{12}$

**d** All the routes are not equally likely

**e** Considering the number of routes available at each step $\frac{1}{12} = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2}$

**8 a** QW, WV, RS

**b** Q and S, repeat RV

**c** $a = 17$, $b = 13$, $c = 9$, $d = 17$

**d i** PQWVURSTP has weight 53, PQWVUSTRP has weight 48,

PQWVRSTUP has weight 52

**ii** Best upper bound is 48

**iii** PQWVUSTSRQP

**e** 27

**f** 38

**9 a** A to F and F to D

**b** Yes, $M + M^2 + M^3 + M^4$ has no 0 entries.

**c** No, it is not possible to move between A and D in exactly 4 steps.

**10 a** The matrix is $\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$ A1

**b** The graph is symmetric, hence the steady state is $(1/3, 1/3, 1/3)$ R1

**c** The matrix now is $\begin{pmatrix} 1/2 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/3 & 1/2 \end{pmatrix}$ A1

**d** We solve $\begin{pmatrix} 1/2 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/3 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{array}{c} -x/2 + 2y/3 = 0 \\ x/2 - y + z/3 = 0 \\ y/2 - z/2 = 0 \end{array}$ which gives

$y = 3x/4$, $z = 3x/4$ and therefore the steady state is $(4/10, 3/10, 3/10)$ M1A1

**e** The probability is $\frac{7}{10} \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{3} = \frac{7}{180}$. A1

**11 a** We will use Prim's algorithm.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| A | 0 | 4 | 16 | 32 | 10 | 11 | 29 |
| B | 4 | 0 | 23 | 10 | 25 | 18 | 20 |
| C | 16 | 23 | 0 | 12 | 14 | 3 | 30 |
| D | 32 | 10 | 12 | 0 | 19 | 13 | 15 |
| E | 10 | 25 | 14 | 19 | 0 | 40 | 43 |
| F | 11 | 18 | 3 | 13 | 40 | 0 | 32 |
| G | 29 | 20 | 30 | 15 | 43 | 32 | 0 |

Therefore, the minimal cost is $4+10+3+12+10+3+15=57$ thousand pounds. M2A1

**b** We start by joining F and G.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| A | 0 | | 16 | 32 | 10 | 11 | 29 |
| B | | 0 | 23 | 10 | 25 | 18 | 20 |
| C | 16 | 23 | 0 | 12 | 14 | 3 | 30 |
| D | 32 | 10 | 12 | 0 | 19 | 13 | 15 |
| E | 10 | 25 | 14 | 19 | 0 | 40 | 43 |
| F | 11 | 18 | 3 | 13 | 40 | 0 | 32 |
| G | 29 | 20 | 30 | 15 | 43 | 32 | 0 |

Now the minimal cost is $10+12+10+14+32+15=93$ thousands of pounds. M2A1

**12 a** The difference between the *k+1*-th and *k*-th points is

**b** The integral is the limit of an infinite sum.

$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{\delta x_i \to 0}\left(\sum_{i=1}^{n} y_i \delta x_i\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{1}{x_i^2}\delta x_i\right)$. The sum can be written as M1A1

$$\sum_{i=1}^{n} \frac{1}{x_i^2}\delta x_i = \frac{1}{n} + \frac{1}{n}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{-2} + \ldots + \frac{1}{n}\left(1+\frac{1}{n}\right)^{k}\left(1+\frac{1}{n}\right)^{-2k} + \ldots$$

$$= \frac{1}{n} + \frac{1}{n}\left(1+\frac{1}{n}\right)^{-1} + \ldots + \frac{1}{n}\left(1+\frac{1}{n}\right)^{-k} + \ldots$$ M1A1

This is an infinite geometric series with ratio $r = \left(1+\frac{1}{n}\right)^{-1} < 1$, and therefore the

sum is given by $\frac{1}{n}\left(\frac{1}{1-r}\right) = \frac{1+n}{n} = 1 + \frac{1}{n}$. In the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$. M1A1

**c** $\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = \lim_{x\to\infty}\left(-\frac{1}{x}\right) - \left(-\frac{1}{1}\right) = 0 + 1 = 1$ M1A1

**13 a** We can make a substitution $\frac{dx}{dt} = u$, so that the first equation reads

$\frac{du}{dt} = -\alpha u \Rightarrow u(t) = Ae^{-\alpha t}$ and therefore $\frac{dx}{dt} = u \Rightarrow \frac{dx}{dt} = Ae^{-\alpha t}$ M1A1

$\Rightarrow x = -\frac{A}{\alpha}e^{-\alpha t} + B$, and initial conditions give $v_{x0} = A$, $0 = -\frac{v_{x0}}{\alpha} + B \Rightarrow B = \frac{v_{x0}}{\alpha}$.

Therefore $x(t) = \frac{v_{x0}}{\alpha}\left(1 - e^{-\alpha t}\right)$. M1A1

The second equation is solved in a similar way. Substituting $\frac{dy}{dt} = v$, we get

$\frac{dv}{dt} = -\beta v - 9.8 \Rightarrow v = Ce^{-\beta t} - 9.8$. Therefore $y = -\frac{C}{\beta}e^{-\beta t} - \frac{9.8}{\beta}t + D$, and initial

conditions give $y(t) = \frac{v_{y0}}{\beta}\left(1 - e^{-\beta t}\right) + \frac{9.8}{\beta^2}\left(1 - e^{-\beta t}\right) - \frac{9.8}{\beta}t$ M1A1

**b** We can simplify to $y(t) = \left(1 - e^{-\alpha t}\right)\left(\frac{v_{y0}}{\alpha} + \frac{9.8}{\alpha^2}\right) - \frac{9.8}{\alpha}t$. A1

From the equation $x(t) = \frac{v_{x0}}{\alpha}\left(1 - e^{-\alpha t}\right)$ we can express $t = \frac{1}{\alpha}\log\left(\frac{v_{x0}}{v_{x0} - \alpha x}\right)$. M1A1

Also, $\left(1 - e^{-\alpha t}\right) = \frac{\alpha x(t)}{v_{x0}}$, therefore $y = \frac{x}{v_{x0}}\left(v_{y0} + \frac{9.8}{\alpha}\right) - \frac{9.8}{\alpha^2}\log\left(\frac{v_{x0}}{v_{x0} - \alpha x}\right)$. M1A1

**c** Substituting the numerical values we get

$y = \frac{x}{10}\left(10 + \frac{9.8}{0.01}\right) - \frac{9.8}{0.01^2}\log\left(\frac{10}{10 - 0.01x}\right)$. Differentiating this expression with

respect to $x$ gives $\frac{dy}{dx} = 1 + \frac{9.8}{10 \times 0.01} - \frac{9.8}{0.01(10 - 0.01x)} = 99 - \frac{980}{10 - 0.01x}$. M1A1

Therefore the $y$ is maximal where $99 - \frac{980}{10 - 0.01x} = 0 \Rightarrow x \approx 10.1$, M1A1

and attains value of $y = \frac{x}{10}\left(10 + \frac{9.8}{0.01}\right) - \frac{9.8}{0.01^2}\log\left(\frac{10}{10 - 0.01x}\right) = 5.07$. A1

**14 a** The table of least distances is M1

| | A | B | C | D | E |
|---|---|---|---|---|---|
| A | 0 | 32 | 68 | 30 | 60 |
| B | 32 | 0 | 40 | 32 | 92 |
| C | 68 | 40 | 0 | 47 | 65 |
| D | 30 | 32 | 47 | 0 | 90 |
| E | 60 | 92 | 65 | 90 | 0 |

Starting at vertex A, we obtain ADBCEA. The total length of this path is $30+40+65+32+60=227$. Starting at vertex B, we get BADCEAB M1A1

which gives $30+32+65+47+92=266$. A1

Next, starting from C, we get CBADAEC giving $30+32+40+90+65=257$ or

CBDAEC giving $60+32+40+30+65=227$. A1

Starting from D gives DABCEAD, $30+32+40+65+90=257$. A1

Starting from E gives EADBCE, $30+40+65+32+60=227$. A1

**b** Deleting vertex A gives minimum spanning tree as ECBD with weight

$65+40+32=137$. The two deleted edges of least weight are $30$ and $32$,

Therefore, the lower bound is $137+30+32=199$. M1A1

Deleting vertex B gives minimum spanning tree as EADC with weight

$60+30+47=137$. The lower bound is $137+32+32=201$. A1

Deleting vertex C gives minimum spanning tree as EADB with weight

$60+30+32=122$. The lower bound is $122+40+47=209$. A1

Deleting vertex D gives minimum spanning tree as EABC with weight

$60+32+40=132$. The lower bound is $132+30+32=194$. A1

Deleting vertex E gives minimum spanning tree as ADBC with weight

$30+32+40=102$. The lower bound is $102+60+65=227$. A1

Therefore the best lower bound is $194$. A1

**c** The true solution starting from A is ADBCEA with weight

$30+32+40+65+60=227$. The upper bound in this case was the true solution.

The true solution starting from D is DBCEAD with weight

$32+40+65+50+30=217$ which is lower than the upper bound but higher

than the lower bound. R1A2